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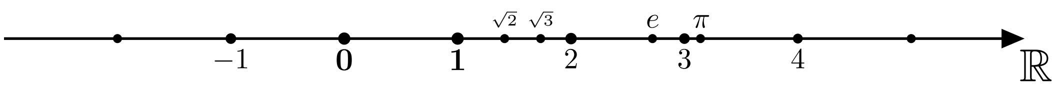
Contents

Chapter 1

Algebra

1.1 Real numbers

1.1.1 Number line

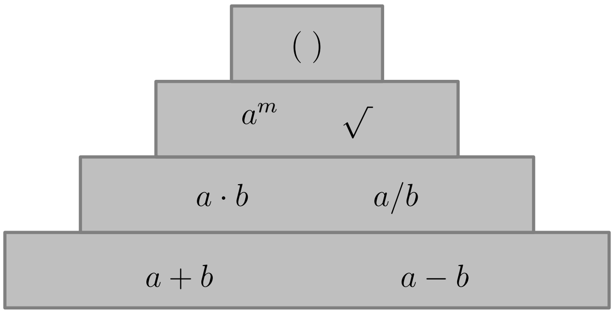


1.1.2 Operator precedence

PEMDAS

Please Excuse My Dear Aunt Sally

- Parentheses
- Exponents
- Multiplication/Division
- Addition/Subtraction.



1.1.3 Parenthesis

$$\begin{aligned} a \cdot (b + c) &= a \cdot b + a \cdot c \\ (a + b) \cdot c &= a \cdot c + b \cdot c \\ (a \cdot b) \cdot c &= a \cdot b \cdot c \end{aligned}$$

1.1.4 Sign

$$\begin{aligned} + \cdot + &= + \\ - \cdot - &= + \\ - \cdot + &= - \\ + \cdot - &= - \end{aligned}$$

1.1.5 Parenthesis and sign

$$\begin{aligned} -(a \cdot b) &= -a \cdot b \\ &\stackrel{\text{or}}{=} a \cdot (-b) \\ +(\pm a \pm b) &= \pm a + \pm b \\ -(\pm a \pm b) &= \mp a \mp b \end{aligned}$$

1.1.6 Fractions

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$
$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$
$$a \cdot \frac{1}{b} = \frac{a}{b}$$
$$a \cdot \frac{b}{c} = \frac{ab}{c}$$
$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$
$$\frac{ac}{bc} = \frac{a}{b}$$
$$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$$
$$\frac{a}{b} \Big/ \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

1.2 Notable products

1.2.1 Notable products

NP1

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

NP2

$$(a+b)(a-b) = a^2 - b^2$$

NP3

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

NP4

$$(a-b)(a^2+ab+b^2) = a^3 - b^3$$
$$(a+b)(a^2-ab+b^2) = a^3 + b^3$$

1.2.2 Newton's binomial

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$\text{met } \binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$$

1.3 Polynomial

1.3.1 Zero polynomial

Form

$$0$$

Remark Degree of the zero polynomial is undefined

1.3.2 First degree polynomial

Form

$$ax + b$$

Root

$$x_0 = \frac{-b}{a}$$

1.3.3 Second degree polynomial

Form

$$ax^2 + bx + c$$

Roots Calculate $D = b^2 - 4ac$

- $D > 0 \Rightarrow$ two different roots $\in \mathbb{R}$

$$x_1 = \frac{-b - \sqrt{D}}{2a}$$

$$x_2 = \frac{-b + \sqrt{D}}{2a}$$

- $D = 0 \Rightarrow$ two equal roots $\in \mathbb{R}$

$$x_1 = \frac{-b}{2a}$$

- $D < 0 \Rightarrow$ no roots $\in \mathbb{R}$

Properties

- Factors: Are x_1 and x_2 roots

$$\Rightarrow ax^2 + bx + c = a(x - x_1)(x - x_2)$$

1.3.4 Polynomials of higher degree

Form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

- x : unknown
- $a_i x^i$: terms
- a_i : coefficients
- n : degree
- a_0 : constant term

Properties

- The **value** of p in $\alpha \in \mathbb{R}$ is

$$p(\alpha) = a_n \alpha^n + a_{n-1} \alpha^{n-1} + \cdots + a_1 \alpha + a_0$$

- $a \in \mathbb{R}$ is a **root** of $p(x)$

$$\Leftrightarrow p(a) = 0$$

  Synonyms: zero value of zero point

- $d(x)$ is a **factor** of $p(x)$

$$\Leftrightarrow p(x) = d(x) \cdot q(x)$$

  with $q(x)$ a polynomial

- Link root and factors

$$p(a) = 0 \Leftrightarrow p(x) = (x - a) \cdot q(x)$$

  with $q(x)$ a polynomial

1.4 Exponents

1.4.1 Definition exponents

$$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}}$$

$$a^1 = a$$

$$a^0 = 1$$

1.4.2 Negative exponent

$$a^{-n} = \frac{1}{a^n}$$

1.4.3 Sign of exponent

$$(\pm a)^{\overbrace{2k}^{\text{even}}} = +a^{2k}$$

$$(\pm a)^{\overbrace{2k+1}^{\text{odd}}} = \pm a^{2k+1}$$

1.4.4 Properties exponents

$$a^m \cdot a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$\left(a^m\right)^n = a^{m \cdot n}$$

$$(ab)^m = a^m b^m$$

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

1.4.5 Rationale exponent

$$a^{n/m} = \sqrt[m]{a^n}$$

1.5 Square roots

1.5.1 Definition square root

x is a **square root** of a

$$\Leftrightarrow x^2 = a$$

1.5.2 Properties square root

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

$$\sqrt{a^m} = \sqrt{a}^m$$

$$\sqrt{a^2} = |a|$$

$$\sqrt{a} = \sqrt[2]{a}$$

1.6 Roots

1.6.1 Definition roots

x is a n**-th root** of a

$$\Leftrightarrow x^n = a$$

1.6.2 Properties roots

$$\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

$$\sqrt[n]{a^m} = \sqrt[n]{a}^m$$

$$\sqrt[n]{\sqrt[m]{a}} = \sqrt[m \cdot n]{a}$$

$$\sqrt[n \cdot p]{a^{m \cdot p}} = \sqrt[n]{a^m}$$

$$\sqrt[n]{a^n} = \begin{cases} |a| & (n \text{ is even}) \\ a & (n \text{ is odd}) \end{cases}$$

1.7 Logarithms

1.7.1 Definition logarithm

$$\log_a x = y \Leftrightarrow a^y = x$$

1.7.2 Notable logarithms

Name	Notation
Briggse logarithm	$\log x = \log_{10} x$
Neperian logarithm	$\ln x = \log_e x$
Binary logarithm	$\lg x = \log_2 x$

1.7.3 Properties logarithms

Same base

$$\log_a(x \cdot y) = \log_a x + \log_a y$$
$$\log_a \frac{x}{y} = \log_a x - \log_a y$$
$$\log_a \frac{1}{y} = -\log_a y$$
$$\log_a(x^p) = p \cdot \log_a x$$

Different base

$$\log_b x = \frac{\log_a x}{\log_a b}$$
$$\log_b x = \frac{1}{\log_x b}$$

1.8 Absolute value

1.8.1 Absolute value

Definition

$$|a| = \begin{cases} a & (a \geq 0) \\ -a & (a \leq 0) \end{cases}$$

Corollary

$$|x| = \sqrt{x^2}$$

1.8.2 Properties absolute value

$$
\begin{aligned}
|a| &\geq 0 \\
|a| &= 0 \Leftrightarrow a = 0 \\
|a \cdot b| &= |a| \cdot |b| \\
|a + b| &\leq |a| + |b| \\
\Big||a|\Big| &= |a| \\
|-a| &= |a| \\
|a - b| &= 0 \Leftrightarrow a = b \\
|a - b| &\leq |a - c| + |c - b| \\
\left|\frac{a}{b}\right| &= \frac{|a|}{|b|} \quad (b \neq 0) \\
|a - b| &\geq \Big||a| - |b|\Big| \\
|a| &\leq b \Leftrightarrow -b \leq a \leq b \\
|a| &\geq b \Leftrightarrow a \leq -b \text{ of } a \geq b
\end{aligned}
$$

1.9 Divisibility

1.9.1 Divisibility

Definition: d divides a

$$
\begin{aligned}
&\Leftrightarrow d|a \\
&\Leftrightarrow d \in \mathbb{Z}_0 \wedge \exists q \in \mathbb{Z} : a = qd
\end{aligned}
$$

- $d \in \mathbb{Z}_0$: divisor of a
- $a \in \mathbb{Z}$: multiple of d
- $\mathrm{del}(a) = \{d|a : d \in \mathbb{Z}_0\}$

Properties:

- $d|a \Rightarrow -d|a$

- $d|a \Rightarrow d|-a$
- $\forall a \in \mathbb{Z} : \pm 1|a$
- $\forall d \in \mathbb{Z}_0 : d|0$
- $\forall a \in \mathbb{Z} : 0 \nmid a$
- $\forall d \in \mathbb{Z}_0 : d|d$

Linear Combinations:

$$\left.\begin{matrix} d|a \\ d|b \end{matrix}\right\} \Rightarrow d|xa + yb,\ \forall x, y \in \mathbb{Z}$$

1.9.2 Common multiples/divisors

Least Common Multiple: m is the smallest natural number that is a multiple of a and b

$$\Leftrightarrow m = \text{lcm(a,b)}$$

$$\Leftrightarrow \begin{cases} m \in a\mathbb{Z} \cap b\mathbb{Z} \\ \forall c \in a\mathbb{Z} \cap b\mathbb{Z} : c \geq m \wedge c \neq 0 \end{cases}$$

Greatest Common Divisor: d is the greatest natural number that is a divisor of a and b

$$\Leftrightarrow d = \text{gcd(a,b)}$$

$$\Leftrightarrow \begin{cases} d \in \text{del}(a) \cap \text{del}(b) \\ \forall c \in \text{del}(a) \cap \text{del}(b) : c \leq d \end{cases}$$

Property:

$$\text{lcm}(a,b) \cdot \text{gcd}(a,b) = a \cdot b$$

1.9.3 Euclidean Division

$$\forall a \in \mathbb{Z}, b \in \mathbb{Z}_0 : \exists! q \in \mathbb{Z}, \exists! r \in \mathbb{Z} :$$
$$a = bq + r \qquad (0 \leq r < |b|)$$

- q : quotient
- r : remainder

1.9.4 Prime Numbers

Definition: A prime number $p \in \mathbb{N} \setminus \{0,1\}$ has only 1 and itself as positive divisors

$$\begin{aligned} &\Leftrightarrow p \text{ is prime} \\ &\Leftrightarrow \operatorname{div}(p) \cap \mathbb{N} = \{1, p\} \end{aligned}$$

Fundamental Theorem of Arithmetic: Every $n \in \mathbb{N} \setminus \{0,1\}$ can be factored into prime factors

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_k^{\alpha_k}$$

- $p_1, ..., p_k$: distinct prime factors
- $\alpha_1, ..., \alpha_k \in \mathbb{N}_0$
- $\#\operatorname{div}(n) = (\alpha_1 + 1)(\alpha_2 + 1)...(\alpha_k + 1)$

1.10 Summations

1.10.1 Definition

$$\sum_{i=0}^{n} a_i = a_0 + a_1 + \cdots + a_n$$

- Σ : summation-symbol
- i : summation-index
- a_i : terms

1.10.2 Operations on summations

Group sums

$$\sum_{i=0}^{n} a_i + \sum_{i=0}^{n} b_i = \sum_{i=0}^{n} (a_i + b_i)$$

Group difference

$$\sum_{i=0}^{n} a_i - \sum_{i=0}^{n} b_i = \sum_{i=0}^{n} (a_i - b_i)$$

Split sum

$$\sum_{i=0}^{n} a_i = \sum_{i=0}^{m} a_i + \sum_{i=m+1}^{n} a_i \qquad (m < n)$$

Scalair product sum

$$\sum_{i=0}^{n} (k \cdot a_i) = k \cdot \sum_{i=0}^{n} a_i$$

Sum of constants

$$\sum_{i=0}^{n} k = (n+1)k$$

1.10.3 Summation identities

Sum of natural numbers

$$\sum_{i=1}^{n} i = \frac{n \cdot (n+1)}{2}$$

Sum of squares

$$\sum_{i=1}^{n} i^2 = \frac{n \cdot (n+1) \cdot (2n+1)}{6}$$

Geometric sum

$$\sum_{i=1}^{n} x^i = \frac{x - x^{n+1}}{1 - x} \qquad \text{met } x \neq 1$$

Telescopic sum

$$\sum_{i=1}^{n} (a_{i+1} - a_i) = a_{n+1} - a_1$$

Chapter 2

Linear algebra

2.1 Matrices

2.1.1 Definition matrix

A $m \times n$-matrix has the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- a_{ij} : elements
- $m \times n$: orde (*size*)
- m : number of rows
- n : number of columns
- Notation: A, B, C, ...
- Short notation: (a_{ij})
- $\mathbb{R}^{m \times n}$: set of all $m \times n$-matrices

2.1.2 Special matrices

Zero matrix $O_{m,n}$ is a $m \times n$-matrix with all $a_{ij} = 0$

Row matrix precisely one row

Column matrix precisely one column

Square matrix equal number of rows as columns, $m = n$.

- **main diagonal** : elements a_{11}, a_{22}, ... , a_{nn}
- **anti-diagonal** : elements $a_{n\,1}$, $a_{n-1\,2}$, ... , $a_{1\,n}$

Diagonal matrix a square matrix $a_{ij} = 0$ with $i \neq j$

Scalar matrix diagonal matrix with $a_{11} = a_{22} = \cdots = a_{nn}$

Upper triangular matrix square matrix with $a_{ij} = 0 \quad (i > j)$

Lower triangular matrix square matrix with $a_{ij} = 0 \quad (i < j)$

Identity matrix I_n of order n with $a_{ii} = 1$

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

2.1.3 Sum two matrices

Definition

$$A, B \in \mathbb{R}^{m \times n} \Rightarrow A + B = C \in \mathbb{R}^{m \times n}$$

with

$$(c_{ij}) = (a_{ij} + b_{ij})$$

Group properties

$$\begin{aligned} A + B &= C \in \mathbb{R}^{m \times n} \\ A + (B + C) &= (A + B) + C \\ A + O_{m,n} &= A = O_{m,n} + A \\ A + (-A) &= O_{m,n} = (-A) + A \\ A + B &= B + A \end{aligned}$$

$$\Rightarrow \mathbb{R}^{m \times n}, + \text{ is a commutative group}$$

2.1.4 Scalar product matrices

Definition

$$\lambda \in \mathbb{R}, A \in \mathbb{R}^{m \times n}$$

$$\Rightarrow \lambda \cdot A = \lambda \cdot \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} \lambda \cdot a_{11} & \lambda \cdot a_{12} & \cdots & \lambda \cdot a_{1n} \\ \lambda \cdot a_{21} & \lambda \cdot a_{22} & \cdots & \lambda \cdot a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda \cdot a_{m1} & \lambda \cdot a_{m2} & \cdots & \lambda \cdot a_{mn} \end{pmatrix}$$

Properties

$$\lambda \cdot A \in \mathbb{R}^{m \times n}$$

$$\lambda \cdot (\mu \cdot A) = (\lambda \mu) \cdot A$$

$$\lambda \cdot (A + B) = \lambda \cdot A + \lambda \cdot B$$

$$(\lambda + \mu) \cdot A = \lambda \cdot A + \mu \cdot A$$

$$1 \cdot A = A$$

$$0 \cdot A = O_{m,n}$$

$$\Rightarrow \mathbb{R}, \mathbb{R}^{m \times n}, + \text{ is a real vector space}$$

2.1.5 Product two matrices

Definition

$$A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{n \times p} \Rightarrow A \cdot B = C \in \mathbb{R}^{m \times p}$$

with

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

Associative

$$A \cdot (B \cdot C) = (A \cdot B) \cdot C$$

Absorbing element

$$A_{m\times n}\cdot O_{n\times p}=O_{m\times p}=O_{m\times n}\cdot B_{n\times p}$$

Neutral element

$$A\cdot I=A=I\cdot A$$

NOT commutative!

2.1.6 Transposed matrix

Definition If $A=(a_{ij})$ then the **transposed matrix** A^T is

$$A^T=(a_{ji})$$

(rows $\leftrightarrow$ columns)

Properties

$$\begin{aligned}(A^T)^T&=A\\(\lambda\cdot A)^T&=\lambda\cdot A^T\\(A+B)^T&=A^T+B^T\\(A\cdot B)^T&=B^T\cdot A^T\end{aligned}$$

2.1.7 Special square matrices

$$\begin{aligned}A\text{ is }\textbf{symmetric}&\Leftrightarrow A^T=A\\A\text{ is }\textbf{anti-symmetric}&\Leftrightarrow A^T=-A\\A\text{ is }\textbf{orthogonal}&\Leftrightarrow A\cdot A^T=I_n=A^T\cdot A\\A\text{ is }\textbf{idempotent}&\Leftrightarrow A^2=A\\A\text{ is }\textbf{nilpotent}\text{ with index }p&\Leftrightarrow p=\min\{n\in\mathbb{N}:A^n=O\}\end{aligned}$$

2.1.8 Inverse matrices

A^{-1} is the **inverse matrix** of A

$$\Leftrightarrow A\cdot A^{-1}=I_n=A^{-1}\cdot A$$

2.2 System of linear equations

2.2.1 Definition

System of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \qquad\qquad \vdots \qquad\qquad\quad = \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

- a_{ij} : coefficients
- x_i : unknowns
- b_i : constant terms (known)

Matrix representation

$$A \cdot X = B$$

- A : coefficient-matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- X : matrix of unknowns

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

- B : matrix of constant terms

$$B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Augmented matrix

$$A_B = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & | & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & | & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & | & b_m \end{pmatrix}$$

2.2.2 Elementary row operations

Swap two rows

$$R_i \leftrightarrow R_j$$

Multiply a row by a nonzero scalar

$$R_i \leftarrow k \cdot R_i \qquad \text{met } k \neq 0$$

Add to one row a multiple of another row

$$R_i \leftarrow R_i + k \cdot R_j \qquad \text{met } i \neq j$$

2.2.3 Echelon form

Reduced row echelon form has

- all zero rows at the bottom
- **pivot**: leading non-zero coefficient in a row is 1
- all elements above and below the pivot are zero
- **echelon form**: each row starts with more zeros than the previous row

$$\begin{pmatrix} 1 & 0 & a_1 & 0 & b_1 \\ 0 & 1 & a_2 & 0 & b_2 \\ 0 & 0 & 0 & 1 & b_3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Rang

- rank(A): number of non-zero rows in a reduced row echelon matrix of A

2.2.4 Homogeneous system

General

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \quad = \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$

$$\Leftrightarrow A \cdot X = 0$$

Homogeneous $2 \times 3-$system

$$\begin{cases} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \end{cases}$$

with rank 2 has infinity many ($\lambda \in \mathbb{R}$) solutions:

$$x = \lambda \cdot \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \quad y = -\lambda \cdot \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \quad z = \lambda \cdot \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

2.3 Determinants

2.3.1 Definitions

Determinant of a 1×1-matrix

$$\begin{vmatrix} a_{11} \end{vmatrix} = a_{11}$$

Determinant of a 2×2-matrix

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Determinant of a 3×3-matrix

Rule of Sarrus

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

Laplace expansion

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3}$$

$$= a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j}$$

- $A_{ij} = (-1)^{i+j} \cdot M_{ij}$: cofactor of a_{ij}, product of $(-1)^{i+j}$ en the minor M_{ij} of a_{ij}
- M_{ij} : minor M_{ij} of a_{ij}, determinant of the submatrix where the i-th row and the j-th column of A is removed

Determinant of a $n \times n$-matrix

$$\begin{aligned} \det(A) &= \sum_{j=1}^{n} a_{ij}A_{ij} \qquad i \text{ a random row} \\ &= \sum_{i=1}^{n} a_{ij}A_{ij} \qquad j \text{ a random column} \end{aligned}$$

2.3.2 Properties of determinants

Main properties

- Switch two rows or columns $\Rightarrow$ change sign of determinant
- Two equals rows or columns $\Rightarrow$ determinant $= 0$
- A row/column is the multiple of an other row/column $\Rightarrow$ determinant $= 0$
- Row/column times a factor k $\Rightarrow$ determinant times factor k
- Add to a row/column the multiple of an other row/column $\Rightarrow$ determinant stays the same

Calculations with determinants

$$\det(A \cdot B) = \det(A) \cdot \det(B)$$

$$\det(k \cdot A) = k^n \cdot \det(A)$$

$$\det(A^T) = \det(A)$$

Existence of inverse matrix

$$\exists A^{-1} \Leftrightarrow \det(A) \neq 0$$

Calculating the inverse matrix

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)$$

- $\operatorname{adj}(A)$: transpose of the cofactor matrix of A

Cramer's Rule System

$$A \cdot X = B$$

with $\det(A) \neq 0$:

$$x_i = \frac{\det(A_i)}{\det(A)}$$

- A_i: matrix A with column i replaced by B.

Chapter 3

Trigonometry

3.1 Angle

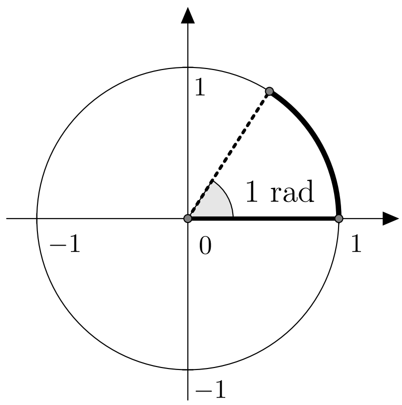
3.1.1 Degree

$$1^\circ = 60'$$
$$1' = 60''$$
$$\Rightarrow 1^\circ = 3600''$$

- $^\circ$: degree, complete circle = 360°
- $'$: minutes
- $''$: seconds

3.1.2 Radian

A **radian** (1 rad): angle subtended from center of a circle where the arc = length of the radius.



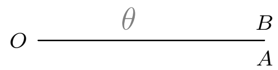
- θ rad $= (\theta + k \cdot 2\pi)$ rad $(k \in \mathbb{Z})$
- symbol *rad* can be omitted

3.1.3 Angles

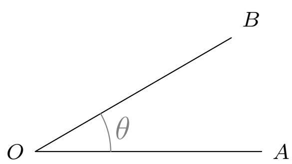
- angle are defined to a multiple of 360°
- positive angles: counter clock wise
- negative angles: clock wise

$\theta \in [0°, 360°[$:

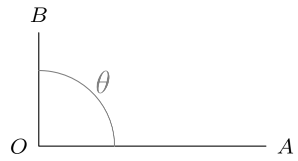
- θ **zero angle** $\Leftrightarrow \theta = 0°$



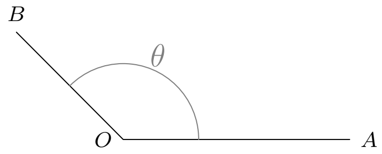
- θ **acute angle** $\Leftrightarrow 0° < \theta < 90°$



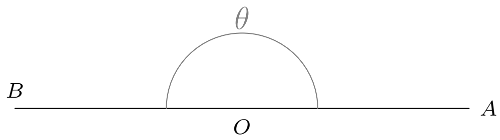
- θ **right angle** $\Leftrightarrow \theta = 90°$



- θ **obtuse angle** $\Leftrightarrow 90^\circ < \theta < 180^\circ$



- θ **straight angle** $\Leftrightarrow \theta = 180^\circ$



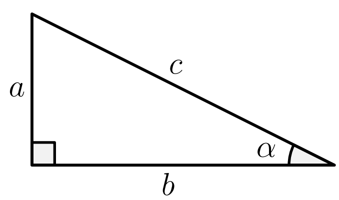
3.1.4 Connection between radians and degrees

$$360^\circ = 2\pi \text{ rad} \qquad \Rightarrow \qquad x^\circ = x \cdot \frac{\pi}{180} \text{ rad}$$

$$x \text{ rad} = x \cdot \frac{180^\circ}{\pi}$$

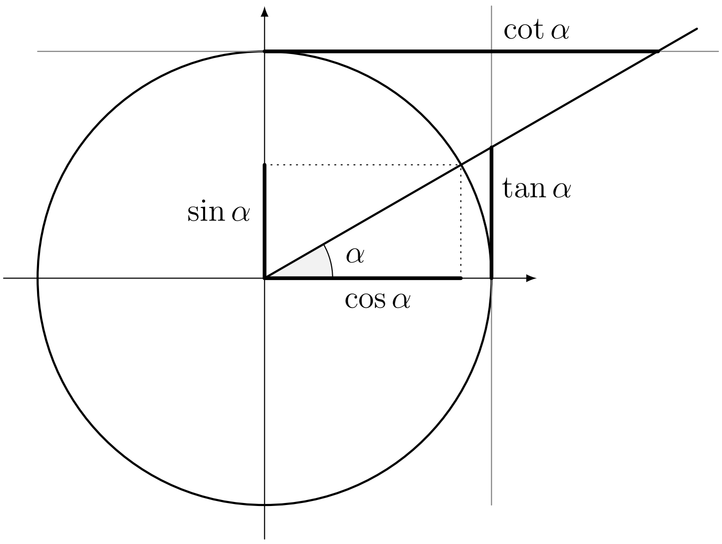
3.2 Trigonometric numbers

3.2.1 Trigonometric numbers in a right triangle



$$\sin\alpha = \frac{a}{c}$$
$$\cos\alpha = \frac{b}{c}$$
$$\tan\alpha = \frac{a}{b}$$

3.2.2 Trigonometric numbers in the unit circle



3.2.3 Definitions

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$$

$$\cot\alpha = \frac{\cos\alpha}{\sin\alpha}$$

$$\sec\alpha = \frac{1}{\cos\alpha}$$

$$\csc\alpha = \frac{1}{\sin\alpha}$$

3.2.4 Pythagorean identities

Identity

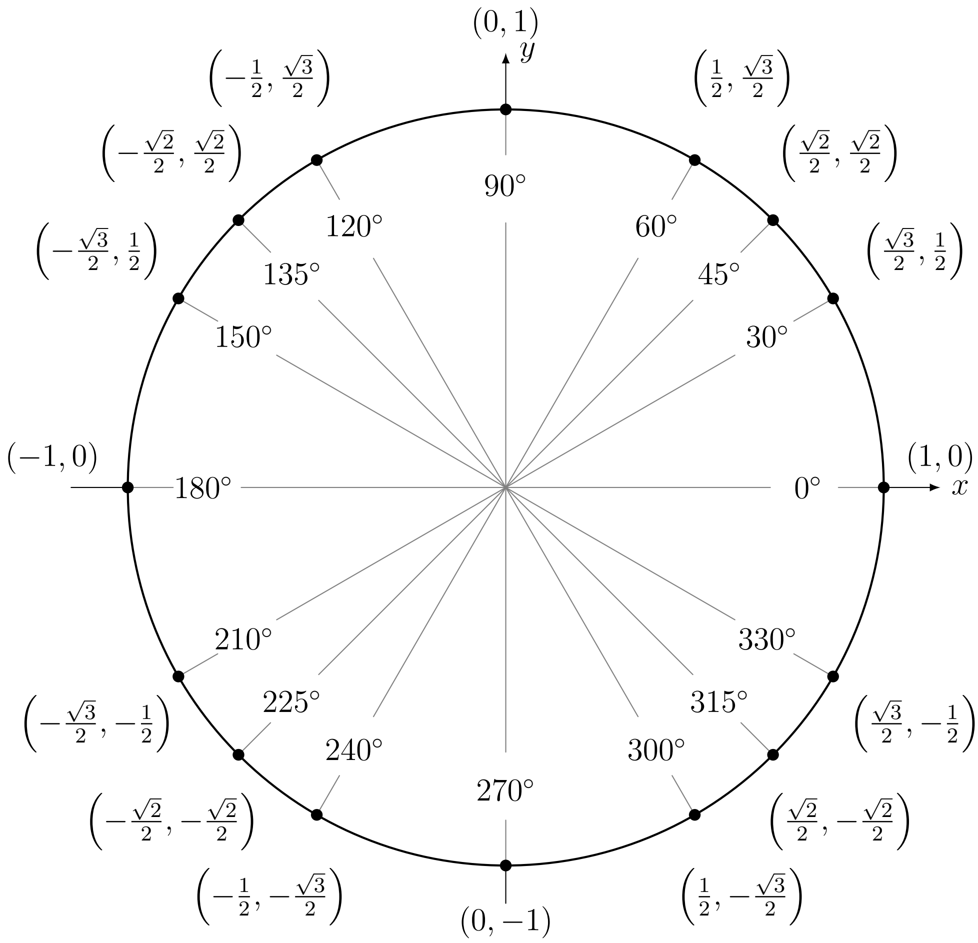
$$\cos^2\alpha + \sin^2\alpha = 1$$

Corrolaries

$$1 + \cot^2\alpha = \csc^2\alpha$$
$$\tan^2\alpha + 1 = \sec^2\alpha$$
$$\sec^2\alpha + \csc^2\alpha = \sec^2\alpha\csc^2\alpha$$

3.2.5 Common trigonometric values

α	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$
$\sin\alpha$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1
$\tan\alpha$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	/	0



3.2.6 Related angles

Angle α

Opposite angles $\beta = -\alpha$

$$
\begin{aligned}
\sin(-\alpha) &= -\sin(\alpha) \\
\cos(-\alpha) &= \cos(\alpha) \\
\tan(-\alpha) &= -\tan(\alpha) \\
\cot(-\alpha) &= -\cot(\alpha)
\end{aligned}
$$

Complementary angles sum angles is 90°

$$\Rightarrow \beta = 90^\circ - \alpha$$

$$\sin(90^\circ - \alpha) = \cos(\alpha)$$
$$\cos(90^\circ - \alpha) = \sin(\alpha)$$
$$\tan(90^\circ - \alpha) = \cot(\alpha)$$
$$\cot(90^\circ - \alpha) = \tan(\alpha)$$

Anti-complementary angles difference angles is 90°

$$\Rightarrow \beta = 90^\circ + \alpha$$

$$\sin(90^\circ + \alpha) = \cos(\alpha)$$
$$\cos(90^\circ + \alpha) = -\sin(\alpha)$$
$$\tan(90^\circ + \alpha) = -\cot(\alpha)$$
$$\cot(90^\circ + \alpha) = -\tan(\alpha)$$

Supplementary angles sum angles is 180°

$$\Rightarrow \beta = 180^\circ - \alpha$$

$$\sin(180^\circ - \alpha) = \sin(\alpha)$$
$$\cos(180^\circ - \alpha) = -\cos(\alpha)$$
$$\tan(180^\circ - \alpha) = -\tan(\alpha)$$
$$\cot(180^\circ - \alpha) = -\cot(\alpha)$$

Anti-supplementary angles difference angles is 180°

$$\Rightarrow \beta = 180^\circ + \alpha$$

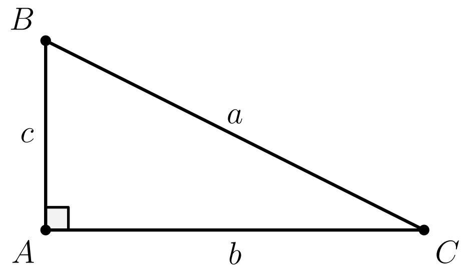
$$\sin(180^\circ + \alpha) = -\sin(\alpha)$$
$$\cos(180^\circ + \alpha) = -\cos(\alpha)$$
$$\tan(180^\circ + \alpha) = \tan(\alpha)$$
$$\cot(180^\circ + \alpha) = \cot(\alpha)$$

3.3 Triangle measurement

Sum in a triangle

$$|\hat{A}| + |\hat{B}| + |\hat{C}| = 180^\circ$$

3.3.1 Right triangles



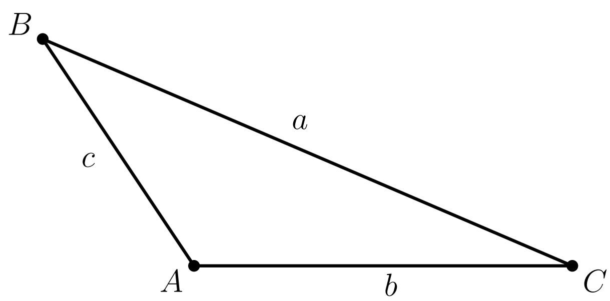
Pythagoras In ΔABC with $|\hat{A}| = 90°$:

$$a^2 = b^2 + c^2$$

Trigonometric ratios

$$\sin|\hat{C}| = \frac{c}{a} = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\cos|\hat{C}| = \frac{b}{a} = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\tan|\hat{C}| = \frac{c}{b} = \frac{\text{opposite}}{\text{adjacent}}$$

3.3.2 Random triangles



Rule of cosines

$$a^2 = b^2 + c^2 - 2 \cdot bc \cdot \cos|\hat{A}|$$
$$b^2 = a^2 + c^2 - 2 \cdot ac \cdot \cos|\hat{B}|$$
$$c^2 = a^2 + b^2 - 2 \cdot ab \cdot \cos|\hat{C}|$$

Sine rule

$$\frac{a}{\sin|\hat{A}|} = \frac{b}{\sin|\hat{B}|} = \frac{c}{\sin|\hat{C}|}$$

3.4 Trigonometric identities

3.4.1 Pythagorean identities

$$\begin{aligned} \cos^2\alpha + \sin^2\alpha = 1 \\ \Rightarrow 1 + \tan^2\alpha = \frac{1}{\cos^2\alpha} \\ \cot^2\alpha + 1 = \frac{1}{\sin^2\alpha} \end{aligned}$$

3.4.2 Angle sum and difference identities

$$\begin{aligned} \sin(\alpha+\beta) &= \sin(\alpha)\cdot\cos(\beta) + \cos(\alpha)\cdot\sin(\beta) \\ \sin(\alpha-\beta) &= \sin(\alpha)\cdot\cos(\beta) - \cos(\alpha)\cdot\sin(\beta) \\ \cos(\alpha+\beta) &= \cos(\alpha)\cdot\cos(\beta) - \sin(\alpha)\cdot\sin(\beta) \\ \cos(\alpha-\beta) &= \cos(\alpha)\cdot\cos(\beta) + \sin(\alpha)\cdot\sin(\beta) \\ \tan(\alpha+\beta) &= \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\cdot\tan(\beta)} \\ \tan(\alpha-\beta) &= \frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\cdot\tan(\beta)} \end{aligned}$$

3.4.3 Double-angle formulae

Times two

$$\begin{aligned}
\sin 2\alpha &= 2 \cdot \sin\alpha \cdot \cos\alpha \\
\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\
\cos 2\alpha &= 1 - 2\sin^2\alpha \\
\cos 2\alpha &= 2\cos^2\alpha - 1 \\
\tan 2\alpha &= \frac{2 \cdot \tan\alpha}{1 - \tan^2\alpha}
\end{aligned}$$

Times three

$$\begin{aligned}
\sin 3\alpha &= 3 \cdot \sin\alpha - 4 \cdot \sin^3\alpha \\
\cos 3\alpha &= 4 \cdot \cos^3\alpha - 3 \cdot \cos\alpha
\end{aligned}$$

3.4.4 Power-reduction formulae

$$\begin{aligned}
\sin^2\alpha &= \frac{1 - \cos 2\alpha}{2} \\
\cos^2\alpha &= \frac{1 + \cos 2\alpha}{2}
\end{aligned}$$

3.4.5 Tangent half-angle substitution

$$\begin{aligned}
\text{Set } t = \tan\frac{\alpha}{2}& \\
\Rightarrow \sin\alpha &= \frac{2t}{1+t^2} \\
\cos\alpha &= \frac{1-t^2}{1+t^2} \\
\tan\alpha &= \frac{2t}{1-t^2}
\end{aligned}$$

3.4.6 Simpson's formulas

Product to sum

$$2 \cdot \sin\alpha \cdot \cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$$
$$2 \cdot \cos\alpha \cdot \sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$$
$$2 \cdot \cos\alpha \cdot \cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$$
$$-2 \cdot \sin\alpha \cdot \sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta)$$

Sum to product

$$\sin p + \sin q = 2 \cdot \sin\frac{p+q}{2} \cdot \cos\frac{p-q}{2}$$
$$\sin p - \sin q = 2 \cdot \cos\frac{p+q}{2} \cdot \sin\frac{p-q}{2}$$
$$\cos p + \cos q = 2 \cdot \cos\frac{p+q}{2} \cdot \cos\frac{p-q}{2}$$
$$\cos p - \cos q = -2 \cdot \sin\frac{p+q}{2} \cdot \sin\frac{p-q}{2}$$

Chapter 4

Geometry

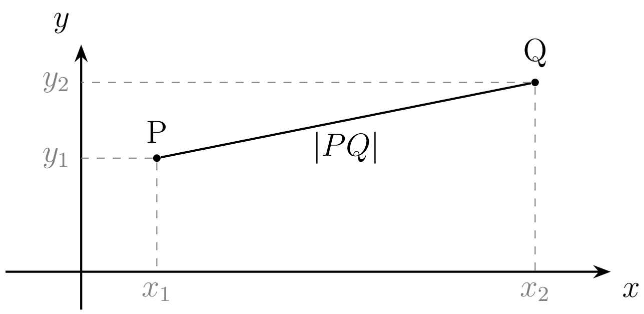
4.1 Plane Geometry

4.1.1 Points

Distance between two points

$$P(x_1, y_1) \wedge Q(x_2, y_2)$$

$$\Rightarrow d(P, Q) = |PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

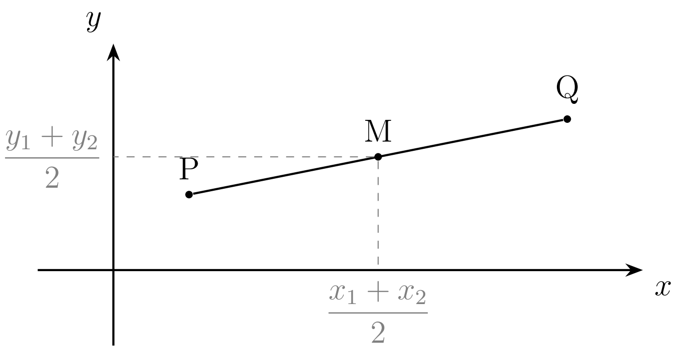


4.1.2 Lines

Midpoint of a line segment

$$P(x_1, y_1) \wedge Q(x_2, y_2) \wedge M = \text{midpoint}_{[PQ]}$$

$$\Rightarrow co(M) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$



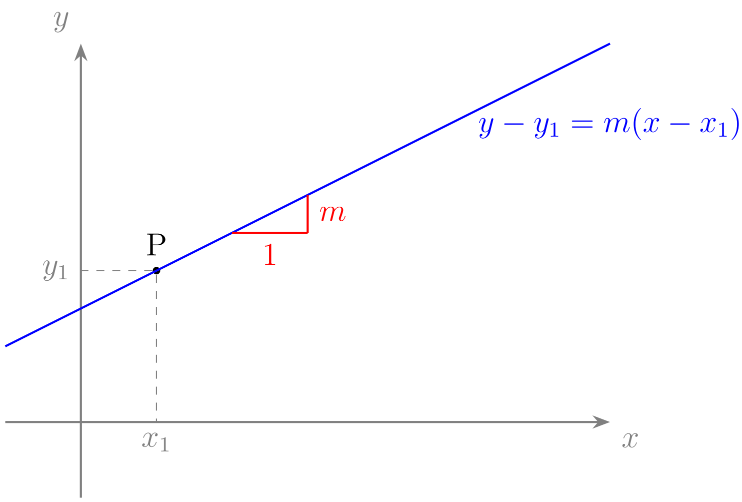
Equation of a line

- Equation of a line passing through point $P(x_1, y_1)$ with slope m:

$$y - y_1 = m(x - x_1)$$

or

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ 1 & m & 0 \end{vmatrix} = 0$$

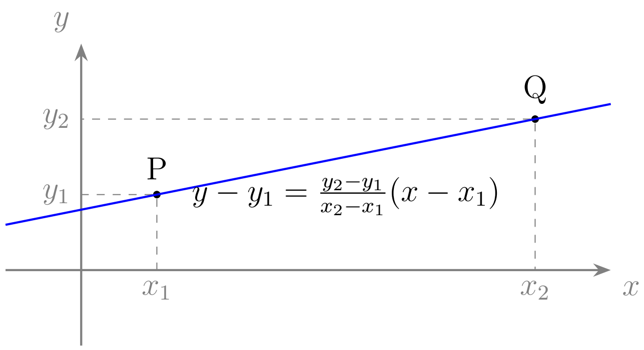


- Equation of a line passing through points $P(x_1, y_1)$ and $Q(x_2, y_2)$:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

or

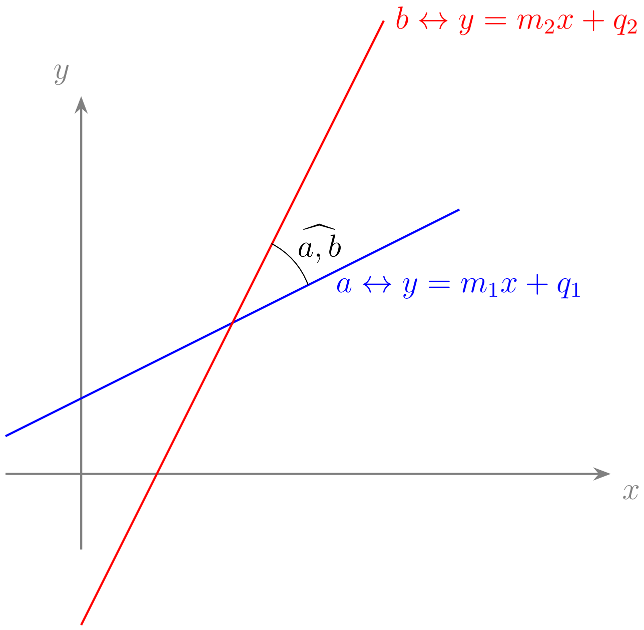
$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$



Relative position of two lines If

$$a \leftrightarrow y = m_1 x + q_1 \wedge b \leftrightarrow y = m_2 x + q_2$$

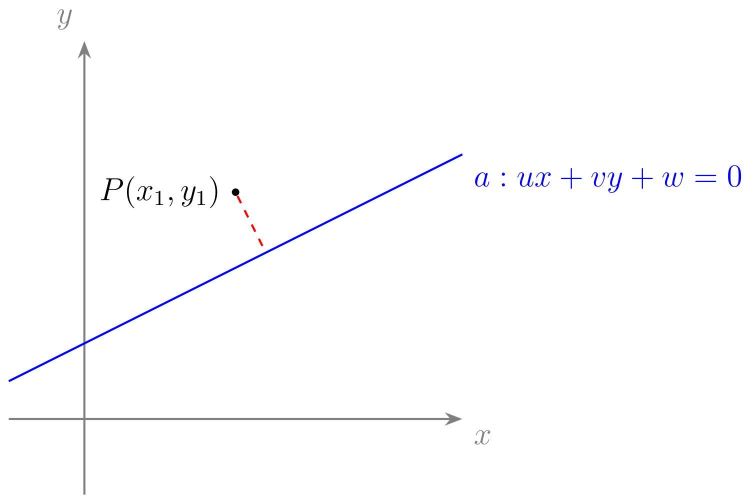
- then the angle between a and b is given by $\cos(\widehat{a,b}) = \dfrac{|1 + m_1 m_2|}{\sqrt{1 + m_1^2}\sqrt{1 + m_2^2}}$
- $a \parallel b$ if $m_1 = m_2$
- $a \perp b$ if $m_1 m_2 = -1$



Distance from a point to a line

$$P(x_1, y_1) \wedge a \leftrightarrow ux + vy + w = 0$$

$$\Rightarrow d(P,a) = \frac{|ux_1 + vy_1 + w|}{\sqrt{u^2+v^2}}$$



4.1.3 Triangles

Centroid of a triangle

$$P(x_1,y_1) \wedge Q(x_2,y_2) \wedge R(x_3,y_3)$$
$$\Rightarrow co(Z) = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

with $Z = \text{centroid}_{\Delta PQR}$

4.1.4 Circles

Definition A circle with center M and radius r:

$$\mathcal{C}(M,r) = \{P \in \pi \mid d(P,M) = r\}$$

Equation of the circle

$$P(x,y) \in \mathcal{C}(M(a,b),r)$$
$$\Leftrightarrow (x-a)^2 + (y-b)^2 = r^2$$

4.2 Plane Figures

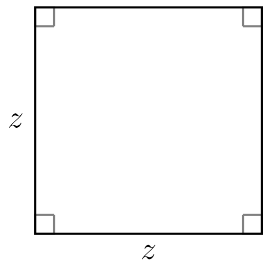
4.2.1 Concepts

Area: The amount of space that the figure occupies.

Perimeter: The total length of the boundary of the figure.

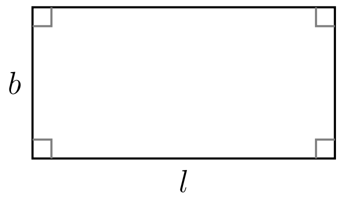
4.2.2 Figures

Square



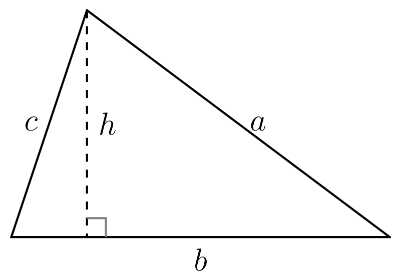
- Figure with four equal sides and four right angles (90°).
- z = side
- Area:
$$A = z^2$$
- Perimeter:
$$P = 4z$$

Rectangle



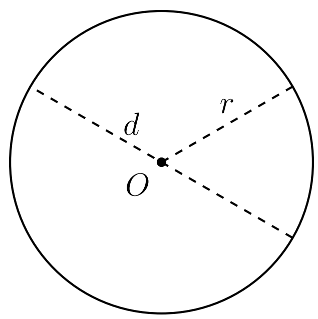
- Figure with two pairs of equal sides and four right angles (90°).
- l = length
- w = width
- Area:
$$A = l \cdot w$$
- Perimeter:
$$P = 2l + 2w$$

Triangle



- Figure with three sides and three angles.
- b = base
- h = height
- Area:
$$A = \frac{1}{2} b \cdot h$$
- Perimeter:
$$P = a + b + c$$
(where a, b, and c are the lengths of the sides)

Circle



- Figure where all points on the boundary are equidistant from the center.
- r = radius
- d = diameter
- O = center
- Area:
$$A = \pi r^2$$

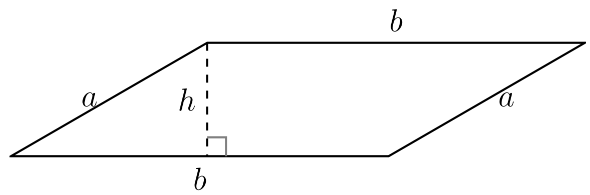
- Perimeter:

$$P = 2\pi r$$

or

$$P = \pi d$$

Parallelogram



- Figure with two pairs of parallel sides.
- b = base
- h = height
- Area:

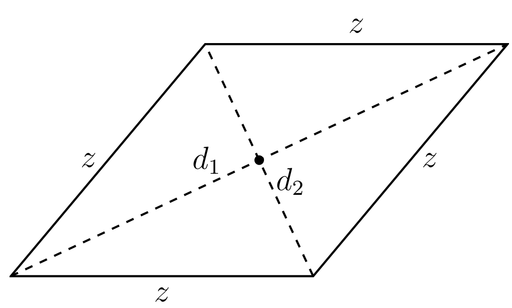
$$A = b \cdot h$$

- Perimeter:

$$P = 2a + 2b$$

(where a and b are the lengths of the sides)

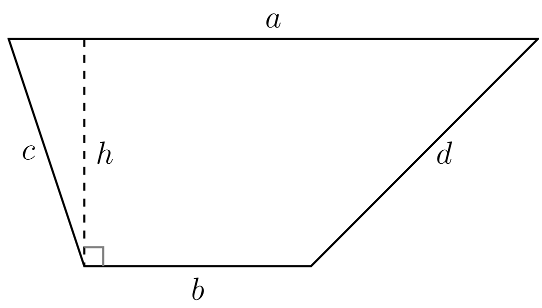
Rhombus



- Figure with four equal sides and opposite angles that are equal.
- d_1 = long diagonal
- d_2 = short diagonal

- Area:
$$A = \frac{1}{2} d_1 \cdot d_2$$
- Perimeter:
$$P = 4z$$
(where z is the side length)

Trapezoid



- Figure with one pair of parallel sides.
- a and b = lengths of the parallel sides
- h = height (distance between the parallel sides)
- Area:
$$A = \frac{1}{2}(a + b) \cdot h$$
- Perimeter:
$$P = a + b + c + d$$
(where c and d are the lengths of the non-parallel sides)

Ellipse

- Figure where the sum of the distances from any point on the boundary to two fixed points (foci) is constant.
- a = semi-major axis (long radius)
- b = semi-minor axis (short radius)
- Area:
$$A = \pi \cdot a \cdot b$$

- Perimeter:
$$P \approx \pi[3(a+b) - \sqrt{(3a+b)(a+3b)}]$$
(approximation)

Kite

- Figure with two pairs of adjacent sides that are equal.
- d_1 = long diagonal
- d_2 = short diagonal
- Area:
$$A = \frac{1}{2} d_1 \cdot d_2$$
- Perimeter:
$$P = 2a + 2b$$
(where a and b are the lengths of the adjacent sides)

Regular Polygon

- Figure with equal sides and angles.
- n = number of sides
- z = side length
- r = apothem (distance from the center to the midpoint of a side)
- Area:
$$A = \frac{1}{2} n \cdot z \cdot r$$
- Perimeter:
$$P = n \cdot z$$

4.3 Synthetic Solid Geometry

4.3.1 Axioms

Basic concepts

symb	name	elements
Σ	space: set of points	$A, B, C, ...$
$\mathcal{R}$	set of lines in Σ	$a, b, c, ...$
$\mathcal{P}$	set of planes in Σ	$\alpha, \beta, \gamma, ...$

Axioms

- *Space is an infinite set of points.*
- *A line is an infinite, non-empty subset of space.*
- *A plane is an infinite, non-empty subset of space.*
- *In every plane of space, the axioms and theorems of plane geometry hold.*
- *Through three non-collinear points A, B and C there passes exactly one plane.*
- *If two distinct points of a line belong to a plane, then this line lies in that plane.*

Consequences

- Through two distinct points there passes at least one plane.
- Two distinct points determine exactly one line.
- Through a line there pass infinitely many planes.

4.3.2 Relative position

Two planes α and β

- Coincident: $\alpha \cap \beta = \alpha = \beta$
- Strictly parallel: $\alpha \cap \beta = \emptyset$
- Intersecting: $\alpha \cap \beta =$ **line of intersection** s

Remark: parallel = coincident or strictly parallel

Line a and plane α

- Contained in: $a \parallel \alpha \wedge a \cap \alpha = a \Leftrightarrow a \subset \alpha$
- Strictly parallel: $a \parallel \alpha \wedge a \cap \alpha = \emptyset$
- Intersecting: $a \cap \alpha =$ **point of intersection** $\{S\}$

Remark: parallel = contained in or strictly parallel

Two lines a and b

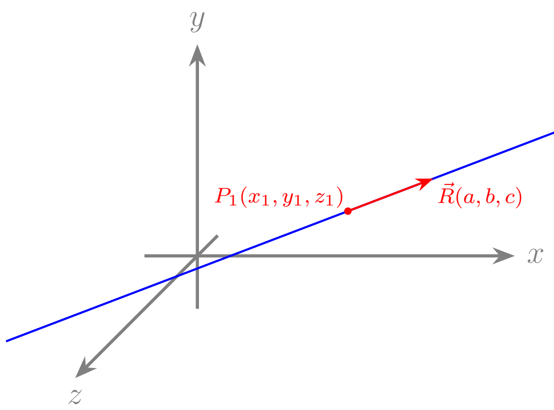
- Coincident: $a \cap b = a = b$
- Intersecting: $a \cap b =$ **point of intersection** $\{S\}$
- Strictly parallel: $a \cap b = \emptyset \wedge a \parallel b$
- Skew: $a \cap b = \emptyset \wedge a \nparallel b$

Remark: parallel = coincident or strictly parallel

4.4 Analytic Solid Geometry

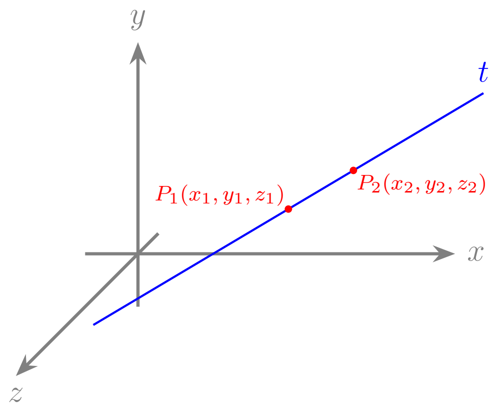
4.4.1 Equations of a line

Line through a point with a direction vector



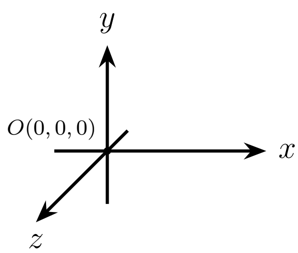
$$
\begin{aligned}
\text{line } t &\leftrightarrow \text{through } P_1(x_1, y_1, z_1) \text{ with direction vector } \vec{R}(a, b, c) \\
&\leftrightarrow \vec{P} = \vec{P_1} + k \cdot \vec{R} \\
&\leftrightarrow \begin{cases} x = x_1 + k \cdot a \\ y = y_1 + k \cdot b \\ z = z_1 + k \cdot c \end{cases} \qquad (k \in \mathbb{R}) \\
&\leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + k \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad (k \in \mathbb{R}) \\
&\leftrightarrow \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \qquad (a, b, c \in \mathbb{R}_0)
\end{aligned}
$$

Line through two points



$$
\begin{aligned}
\text{line } t &\leftrightarrow \text{through } P_1(x_1, y_1, z_1) \text{ and } P_2(x_2, y_2, z_2) \\
&\leftrightarrow \vec{P} = \vec{P_1} + k \cdot (\vec{P_2} - \vec{P_1}) \qquad (k \in \mathbb{R}) \\
&\leftrightarrow \begin{cases} x = x_1 + k \cdot (x_2 - x_1) \\ y = y_1 + k \cdot (y_2 - y_1) \\ z = z_1 + k \cdot (z_2 - z_1) \end{cases} \qquad (k \in \mathbb{R}) \\
&\leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + k \cdot \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix} \qquad (k \in \mathbb{R}) \\
&\leftrightarrow \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \qquad (x_2 \neq x_1, y_2 \neq y_1, z_2 \neq z_1)
\end{aligned}
$$

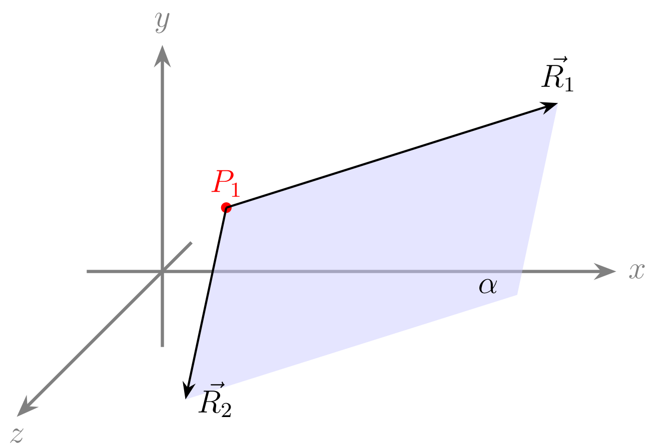
Special lines



$$x\text{-axis} \leftrightarrow y = z = 0$$
$$y\text{-axis} \leftrightarrow x = z = 0$$
$$z\text{-axis} \leftrightarrow x = y = 0$$

4.4.2 Equations of a plane

Plane through a point with two direction vectors



plane $\alpha \leftrightarrow$ through $P_1(x_1, y_1, z_1)$ with direction vectors $\vec{R_1}(a_1, b_1, c_1)$ and $\vec{R_2}(a_2, b_2, c_2)$

$$\leftrightarrow \vec{P} = \vec{P_1} + k \cdot \vec{R_1} + l \cdot \vec{R_2} \qquad (k, l \in \mathbb{R})$$

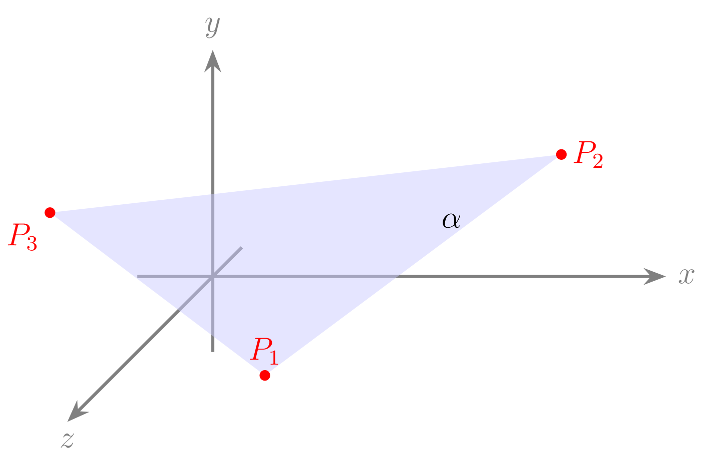
$$\leftrightarrow \begin{cases} x = x_1 + k \cdot a_1 + l \cdot a_2 \\ y = y_1 + k \cdot b_1 + l \cdot b_2 \\ z = z_1 + k \cdot c_1 + l \cdot c_2 \end{cases} \qquad (k, l \in \mathbb{R})$$

$$\leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + k \cdot \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} + l \cdot \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \qquad (k, l \in \mathbb{R})$$

$$\leftrightarrow \begin{vmatrix} x - x_1 & a_1 & a_2 \\ y - y_1 & b_1 & b_2 \\ z - z_1 & c_1 & c_2 \end{vmatrix} = 0$$

$$\leftrightarrow \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \end{vmatrix} = 0$$

Plane through three points



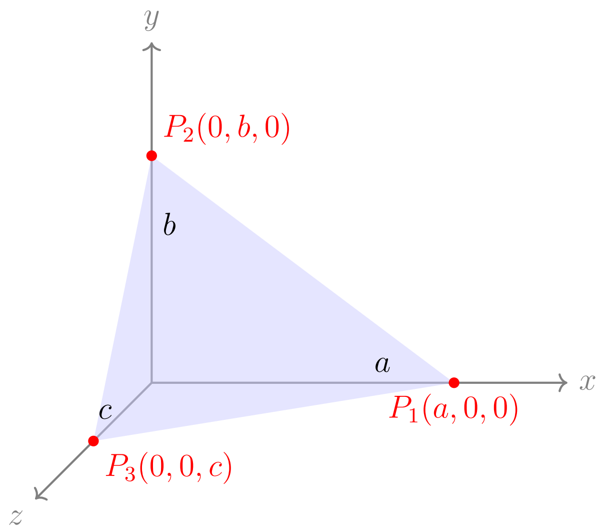
$$
\begin{aligned}
\text{plane } \alpha &\leftrightarrow \text{through } P_1(x_1,y_1,z_1),\ P_2(x_2,y_2,z_2) \text{ and } P_3(x_3,y_3,z_3) \\
&\leftrightarrow \vec{P} = \vec{P_1} + k\cdot(\vec{P_2}-\vec{P_1}) + l\cdot(\vec{P_3}-\vec{P_1}) \qquad (k,l\in\mathbb{R}) \\
&\leftrightarrow \begin{cases} x = x_1 + k\cdot(x_2-x_1) + l\cdot(x_3-x_1) \\ y = y_1 + k\cdot(y_2-y_1) + l\cdot(y_3-y_1) \\ z = z_1 + k\cdot(z_2-z_1) + l\cdot(z_3-z_1) \end{cases} \qquad (k,l\in\mathbb{R}) \\
&\leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + k\cdot\begin{pmatrix} x_2-x_1 \\ y_2-y_1 \\ z_2-z_1 \end{pmatrix} + l\cdot\begin{pmatrix} x_3-x_1 \\ y_3-y_1 \\ z_3-z_1 \end{pmatrix} \qquad (k,l\in\mathbb{R}) \\
&\leftrightarrow \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0
\end{aligned}
$$

General form of a plane

$$\text{plane } \alpha \leftrightarrow ux + vy + wz + t = 0$$

- u, v, w: direction coefficients of the plane
- $\vec{n} = (u, v, w)$: normal vector, $\vec{n} \perp \alpha$
- t: constant term, position relative to $\vec{O}$
- $t = 0 \Leftrightarrow \vec{O} \in \alpha \Leftrightarrow \alpha = \alpha_O$

Plane in intercept form



$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

- a: length of the intercept on the x-axis
- b: length of the intercept on the y-axis
- c: length of the intercept on the z-axis
- a, b and c: the intercepts

Special planes

- xy-plane: $z = 0$
- xz-plane: $y = 0$
- yz-plane: $x = 0$

4.4.3 Relative position of lines

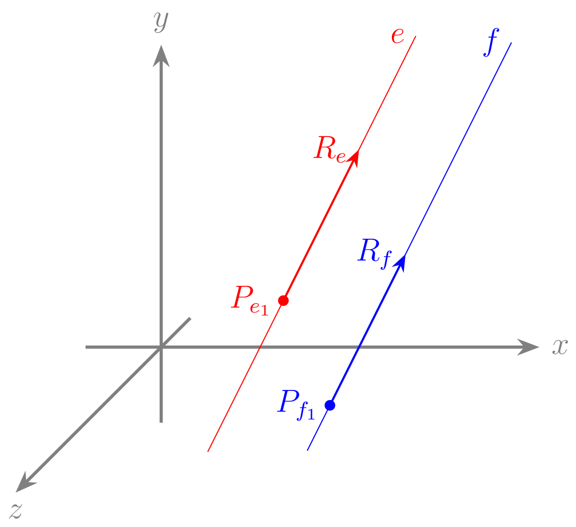
The lines

$$e \leftrightarrow \vec{P_e} = \vec{P_{e_1}} + k \cdot \vec{R_e}$$
$$f \leftrightarrow \vec{P_f} = \vec{P_{f_1}} + l \cdot \vec{R_f}$$

with

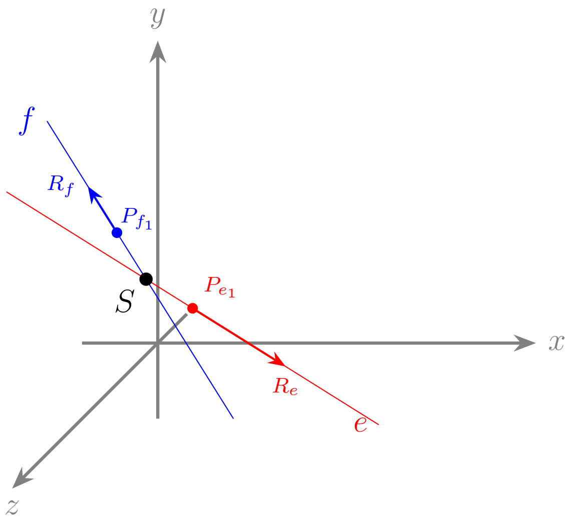
$$P_{e_1}(x_1, y_1, z_1), R_e(a_1, b_1, c_1)$$
$$P_{f_1}(x_2, y_2, z_2), R_f(a_2, b_2, c_2)$$

Parallel



$$\begin{aligned} e \parallel f &\Leftrightarrow \text{direction ratios are proportional} \\ &\Leftrightarrow \exists k \in \mathbb{R} : (a_1, b_1, c_1) = k \cdot (a_2, b_2, c_2) \\ &\Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \end{aligned}$$

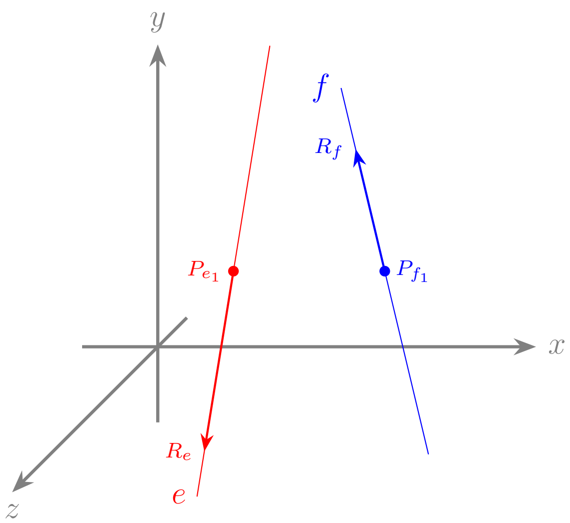
Intersecting



$$e \text{ intersects } f \Leftrightarrow \exists! \text{ solution } (k,l) : P_e = P_f$$

$$\Leftrightarrow \begin{cases} (a_1, b_1, c_1) \nsim (a_2, b_2, c_2) \\ \begin{vmatrix} a_1 & a_2 & x_2 - x_1 \\ b_1 & b_2 & y_2 - y_1 \\ c_1 & c_2 & z_2 - z_1 \end{vmatrix} = 0 \end{cases}$$

Skew



$$e \text{ is skew to } f \Leftrightarrow \nexists \text{ solution } (k,l) : P_e = P_f$$
$$\Leftrightarrow \begin{cases} (a_1,b_1,c_1) \not\sim (a_2,b_2,c_2) \\ \begin{vmatrix} a_1 & a_2 & x_2-x_1 \\ b_1 & b_2 & y_2-y_1 \\ c_1 & c_2 & z_2-z_1 \end{vmatrix} \neq 0 \end{cases}$$

4.4.4 Relative position of a line and a plane

$$\text{line } e \leftrightarrow \begin{cases} x = x_1 + k \cdot a \\ y = y_1 + k \cdot b \\ z = z_1 + k \cdot c \end{cases} \qquad (k \in \mathbb{R})$$
$$\text{plane } \alpha \leftrightarrow ux + vy + wz + t = 0$$

Intersecting

$$e \text{ intersects } \alpha \Leftrightarrow ua + vb + wc \neq 0$$

Parallel

$$e \parallel \alpha \Leftrightarrow ua + vb + wc = 0$$

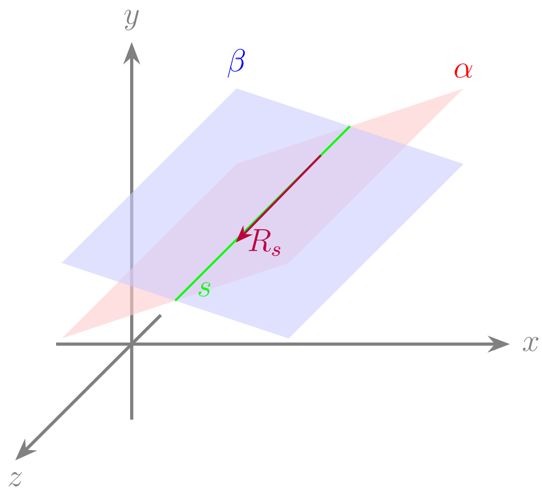
Line contained in the plane

$$e \subset \alpha \Leftrightarrow ua + vb + wc = 0 \land ux_1 + vy_1 + wz_1 + t = 0$$

4.4.5 Relative position of planes

$$\text{plane } \alpha \leftrightarrow u_1x + v_1y + w_1z + t_1 = 0$$
$$\text{plane } \beta \leftrightarrow u_2x + v_2y + w_2z + t_2 = 0$$

Intersecting



$$\alpha \text{ intersects } \beta \Leftrightarrow (u_1, v_1, w_1) \not\sim (u_2, v_2, w_2)$$

- line of intersection: $s = \alpha \cap \beta$:
- direction ratios of the line of intersection: $R_s(a, b, c)$:

$$a = \lambda \cdot \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix}$$

$$b = -\lambda \cdot \begin{vmatrix} u_1 & w_1 \\ u_2 & w_2 \end{vmatrix} \qquad \lambda \text{ arbitrary } \in \mathbb{R}_0$$

$$c = \lambda \cdot \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}$$

Parallel

$$\alpha \parallel \beta \Leftrightarrow (u_1, v_1, w_1) \sim (u_2, v_2, w_2)$$

Coincident

$$\alpha = \beta \Leftrightarrow (u_1, v_1, w_1, t_1) \sim (u_2, v_2, w_2, t_2)$$

4.5 Solids

4.5.1 Concepts

Volume: Amount of space occupied by the figure.

Surface area: Sum of the surfaces of all side faces.

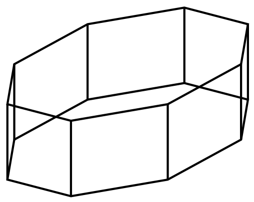
Net: 2D representation by unfolding the side faces and laying them flat.

4.5.2 Figures

Polyhedron

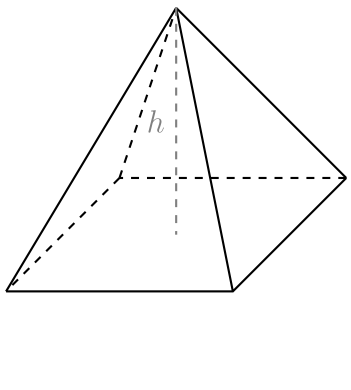
3D figure with polygonal surfaces

Prism



- Polyhedron with parallel, congruent bases connected by rectangular side faces.
- B = base area
- h = height
- P = base perimeter
- Volume:
$$V = B \cdot h$$
- Surface area:
$$A = 2B + P \cdot h$$

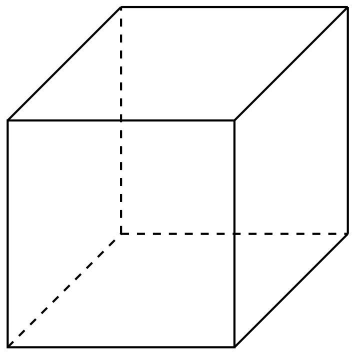
Pyramid



- Polyhedron with a polygonal base and triangular faces that meet at a vertex.
- B = base area
- h = height
- Volume:

$$V = \frac{1}{3}B \cdot h$$

Cube



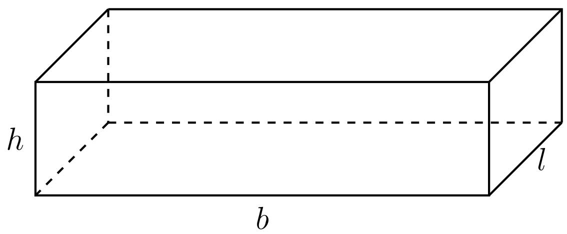
- Prism with six square faces, equal sides, and 90° angles.
- s = side
- Volume:

$$V = s^3$$

- Surface area:

$$A = 6s^2$$

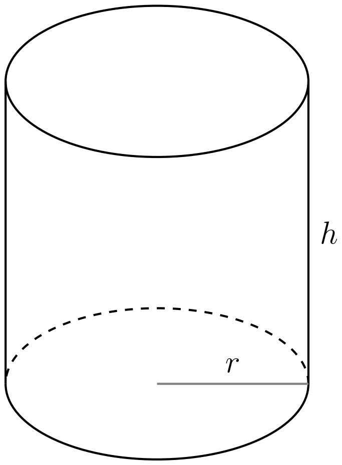
Rectangular Prism



- Prism with six rectangular faces.

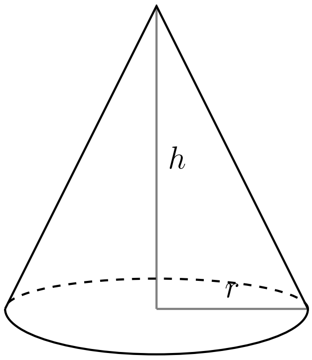
- l = length
- w = width
- h = height
- Volume:
$$V = l \cdot w \cdot h$$
- Surface area:
$$A = 2lw + 2lh + 2wh$$

Cylinder



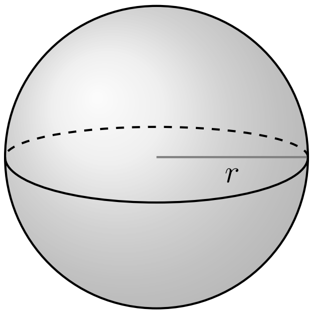
- 3D figure with congruent circular bases connected by a curved surface.
- r = radius
- h = height
- Volume:
$$V = \pi r^2 h$$
- Surface area:
$$A = 2\pi r(r + h)$$

Cone



- 3D figure with a circular base and a curved surface that narrows to a point.
- r = radius
- h = height
- l = slant height
- Volume:
$$V = \frac{1}{3}\pi r^2 h$$
- Surface area:
$$A = \pi r(r + l)$$

Sphere



- 3D figure with all points at equal distance from the center.
- r = radius
- Volume:
$$V = \frac{4}{3}\pi r^3$$

- Surface area:

$$A = 4\pi r^2$$

4.6 Vectors

4.6.1 Definitions of vectors

Bound vector $\vec{v} = \vec{AB}$ a line segment with

- a starting point A
- an end point B
- a length $|AB|$
- a direction, line AB
- a sense, $A \to B$

Free vector a collection of line segments with

- the same length
- the same direction
- the same sense

Representative one arbitrary line segment from the free vector

Position vector $\vec{A} = \vec{OA}$ with O being the origin

Zero vector $\vec{o}:$ length 0

Notation: $\vec{u}, \vec{v}, ...$

Collection: $\mathcal{V}$

Coordinate $co(\vec{A}) = co(A) = (x, y, z)$

Norm of $\vec{v} = \vec{AB}$ is the distance between A and B

$$\|\vec{v}\| = |AB|$$

4.6.2 Adding vectors

Chasles-Möbius formula

$$\vec{AB} + \vec{BC} = \vec{AC}$$

With coordinates

$$\left.\begin{aligned} co(\vec{u}) &= (x_1, y_1, z_1) \\ co(\vec{v}) &= (x_2, y_2, z_2) \end{aligned}\right\} \Rightarrow$$

$$co(\vec{u} + \vec{v}) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

Properties

- I: $\forall \vec{u}, \vec{v} \in \mathcal{V} : \vec{u} + \vec{v} \in \mathcal{V}$
- A: $\forall \vec{u}, \vec{v}, \vec{w} \in \mathcal{V} : (\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
- N: $\exists \vec{o} \in \mathcal{V}, \forall \vec{u} \in \mathcal{V} : \vec{u} + \vec{o} = \vec{u} = \vec{o} + \vec{u}$
- S: $\forall \vec{u} \in \mathcal{V}, \exists! \vec{-u} \in \mathcal{V} : \vec{u} + (\vec{-u}) = \vec{o} = (\vec{-u}) + \vec{u}$
- C: $\forall \vec{u}, \vec{v} \in \mathcal{V} : \vec{u} + \vec{v} = \vec{v} + \vec{u}$

$$\Rightarrow \mathcal{V}, + \text{ is a commutative group}$$

4.6.3 Scalar multiplication of vectors

Definition

$$r \cdot \vec{v} (r \in \mathbb{R}_0, \vec{v} \in \mathcal{V}_o)$$

is a vector with

- length equal to $|r| \cdot \|\vec{v}\|$
- direction as $\vec{v}$
- sense $\begin{cases} \text{as } \vec{v} & (r > 0) \\ \text{opposite } \vec{v} & (r < 0) \end{cases}$

$$0 \cdot \vec{v} = \vec{o}$$

$$r \cdot \vec{o} = \vec{o}$$

With coordinates

$co(\vec{v}) = (x, y, z) \Rightarrow$
$$co(r\vec{v}) = (rx, ry, rz)$$

Properties

- $\forall r \in \mathbb{R}, \forall \vec{v} \in \mathcal{V} : r \cdot \vec{v} \in \mathcal{V}$
- $\forall r, s \in \mathbb{R}, \forall \vec{v} \in \mathcal{V} : r \cdot (s \cdot \vec{v}) = (r \cdot s) \cdot \vec{v}$
- $\forall r \in \mathbb{R}, \forall \vec{u}, \vec{v} \in \mathcal{V} : r \cdot (\vec{u} + \vec{v}) = r \cdot \vec{u} + r \cdot \vec{v}$
- $\forall r, s \in \mathbb{R}, \forall \vec{v} \in \mathcal{V} : (r + s) \cdot \vec{v} = r \cdot \vec{v} + s \cdot \vec{v}$
- $\exists 1 \in \mathbb{R}, \forall \vec{v} \in \mathcal{V} : 1 \cdot \vec{v} = \vec{v}$

4.6.4 Norm of a Vector

Definition

norm of $\vec{v} = \vec{AB}$ is the distance between A and B
$$\|\vec{v}\| = d(A, B) = |AB|$$

With Coordinates

$co(\vec{v}) = (x, y, z) \Rightarrow$
$$\|\vec{v}\| = \sqrt{x^2 + y^2 + z^2}$$

Minkowski Inequality
$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

4.6.5 Dot Product of Two Vectors

Definition

$\vec{u}, \vec{v} \in \mathcal{V}_0$ and ϕ is the angle between $\vec{u}$ and $\vec{v} \Rightarrow$
$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos\phi$$

With Coordinates

$$\left.\begin{aligned} co(\vec{u}) &= (x_1, y_1, z_1) \\ co(\vec{v}) &= (x_2, y_2, z_2) \end{aligned}\right\} \Rightarrow$$

$$\vec{u} \cdot \vec{v} = x_1 \cdot x_2 + y_1 \cdot y_2 + z_1 \cdot z_2$$

Chapter 5

Real functions

5.1 Real Functions

5.1.1 Definitions

Real function: a collection of *pairs* of real numbers (x, y), such that each number x has at *most one image* y.

Notation: f, g, h, f_1, f_2, ...

Terminology: $(x, y) \in \mathbb{R}^2$

- x argument, independent variable, abscissa, x-value
- y image, dependent variable, ordinate, y-value
- $\in$ element of
- $\mathbb{R}^2$ set of all pairs of real numbers

Representations:

- function rule: $f : \mathbb{R} \to \mathbb{R} : x \mapsto f(x)$
- function value table
- graph

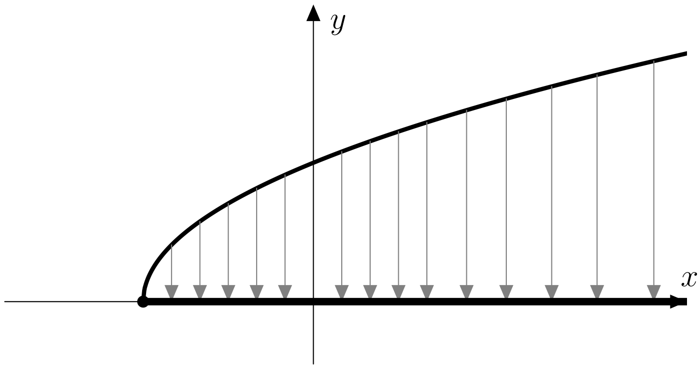
5.1.2 Characteristics of Real Functions

Domain

Definition: the set of all x-values for which the function has an image.

Notation: dom f

Graphically:

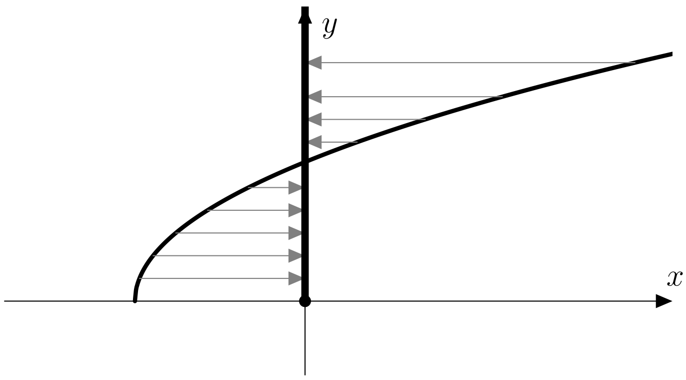


Range

Definition: the set of all images of a function.

Notation: ran f

Graphically:



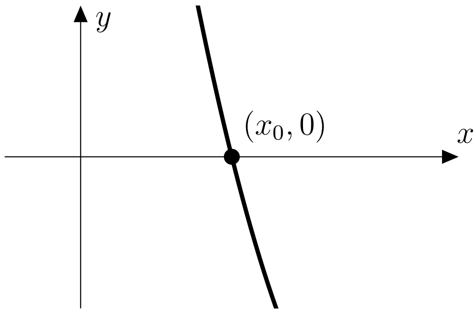
Zeroes

Definition: an x-value for which the image is equal to zero.

Algebraic: Set $y = 0$ in $y = f(x)$. So search for all x for which

$$f(x) = 0$$

Graphically:



Sign Behavior

Definition: determine the sign (positive, negative, or zero) of the image y as the argument x goes through the domain.

Method: Sign diagram with $+$, $-$, 0 and $|$

Increasing and Decreasing

Definitions:

- f is **increasing** in an interval I if for the x-values $a, b \in I$ with $a < b$ it holds that
$$f(a) \leq f(b)$$
- f is **decreasing** in an interval I if for the x-values $a, b \in I$ with $a < b$ it holds that
$$f(a) \geq f(b)$$
- f that is both increasing and decreasing in an interval I is called **constant** on that interval I

Method: Diagram with $\nearrow$ and $\searrow$

Extrema

Definitions:

- real number m is the **minimum** of f on an interval I if for all $x \in I$:
$$f(x) \geq m\ .$$
- real number M is the **maximum** of f on an interval I if for all $x \in I$:
$$f(x) \leq M\ .$$

Method: Diagram with $\nearrow$ and $\searrow$

5.2 Polynomials

5.2.1 Polynomial

A **polynomial** is a sum of monomials:

$$\begin{aligned} P(x) &= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \\ &= \sum_{i=0}^{n} a_i x^i \end{aligned}$$

- x: variable
- $a_i \in \mathbb{R}$: coefficients
- **degree** $deg(P(x))$: the highest exponent of x in P
- **value** of $P(x)$ for a given a: replace x with a
- **root:** all a for which $P(a) = 0$

5.2.2 Degree after operations

$$deg(A(x) + B(x)) \leq \max(deg(A(x)), deg(B(x)))$$
$$deg(A(x) \cdot B(x)) = deg(A(x)) + deg(B(x))$$

5.2.3 Euclidean Division

$$A(x) = D(x) \cdot Q(x) + R(x)$$
$$\text{with } deg(R(x)) < deg(D(x)) \vee R(x) = 0$$

- $A(x)$: dividend
- $D(x) \neq 0$: divisor
- $Q(x)$: quotient
- $R(x)$: remainder
- **exact division:** $R(x) = 0$

- **non-exact division:** $R(x) \neq 0$
- $deg(A(x)) < deg(D(x)) \quad \Rightarrow \quad Q(x) = 0 \wedge R(x) = A(x)$
- $Q(x)$ and $R(x)$ exist and are unique

5.3 Polynomial Functions

5.3.1 General

Polynomial function: a real function with function formula

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Degree: n-th degree

5.3.2 Polynomial Functions of Degree 0

- Synonym: **constant functions**
- Form:
$$f(x) = a \qquad (a \in \mathbb{R})$$
- Graph: horizontal line parallel to the x-axis.
- Domain: $\mathbb{R}$
- Range: $\{a\}$
- Intersection with y-axis: $(0, a)$
- Zeros:
$$\begin{cases} \text{none} & (a \neq 0) \\ \text{every } x \in \mathbb{R} & (a = 0) \end{cases}$$
- Sign diagram:

x	$-\infty$	$+\infty$
$f(x)$	sign of a	

5.3.3 Polynomial Functions of Degree 1

- Synonyms: **first-degree functions**, linear functions
- Form:
$$f(x) = ax + b \qquad (a \in \mathbb{R}_0, b \in \mathbb{R})$$
- Graph: oblique line
$$\begin{cases} \text{increasing oblique line} \nearrow & (a > 0) \\ \text{decreasing oblique line} \searrow & (a < 0) \end{cases}$$
- Gradient: a slope / rate of change
- Domain: $\mathbb{R}$
- Range: $\mathbb{R}$
- Zero: $x_0 = -\frac{b}{a}$
- Intersection with x-axis: $(-\frac{b}{a}, 0)$.
- Intersection with y-axis: $(0, b)$
- Sign diagram:

x	$-\infty$		$-\frac{b}{a}$		$+\infty$
$f(x)$		opposite sign a	0	sign a	

- Extrema: none

5.3.4 Polynomial Functions of Degree 2

- Synonyms: **second-degree functions**, quadratic functions
- Form:
$$f(x) = ax^2 + bx + c \qquad (a \in \mathbb{R}_0, b, c \in \mathbb{R})$$
- Domain: $\mathbb{R}$.

- Range:

$$\begin{cases} \text{ran } f = [f(-\frac{b}{2a}), +\infty) & (a > 0) \\ \text{ran } f = (-\infty, f(-\frac{b}{2a})] & (a < 0) \end{cases}$$

- Zeros: $ax^2 + bx + c = 0$
  - $D = b^2 - 4ac$
  - $D > 0 \Rightarrow$ two distinct zeros

$$x_1 = \frac{-b - \sqrt{D}}{2a} \qquad x_2 = \frac{-b + \sqrt{D}}{2a}$$

  - $D = 0 \Rightarrow$ one zero

$$x_0 = -\frac{b}{2a}$$

  - $D < 0 \Rightarrow$ no zeros

- Sign diagram:
  - $D > 0$

x		x_1		x_2	
$f(x)$	sign a	0	opposite sign a	0	sign a

  - $D = 0$

x	$-\infty$		x_0		$+\infty$
$f(x)$		sign a	0	sign a	

  - $D < 0$

x	$-\infty$	$+\infty$
$f(x)$	sign a	

- Graph: parabola with symmetry axis parallel to the y-axis
  - $a > 0 \Rightarrow$ **upward-opening parabola** (concave up / happy)
  - $a < 0 \Rightarrow$ **downward-opening parabola** (concave down / unhappy)

- Vertex:

$$\begin{aligned} &(-\frac{b}{2a}, f(-\frac{b}{2a})) \\ &= (-\frac{b}{2a}, -\frac{D}{4a}) \end{aligned}$$

  with $D = b^2 - 4ac$

- Increasing and decreasing:
  - $a < 0$

x		$-\dfrac{b}{2a}$	
$f(x)$	$\nearrow$	max $\underset{=f(-\frac{b}{2a})}{}$	$\searrow$

  - $a > 0$

x		$-\dfrac{b}{2a}$	
$f(x)$	$\searrow$	min $\underset{=f(-\frac{b}{2a})}{}$	$\nearrow$

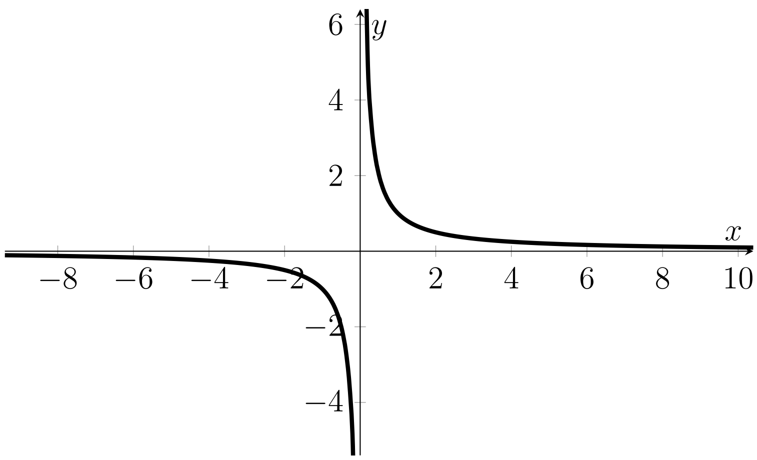
5.4 Rational functions

5.4.1 Basic function

Form:

$$f(x) = \frac{1}{x}$$

Graph: Hyperbola



Domain:

$$\text{dom } f = \mathbb{R}_0$$

Range:

$$\text{ran } f = \mathbb{R}_0$$

Zeros:

$$\text{zeros } f : \diagup$$

Asymptotes:

$$\text{H.A.} : y = 0$$

$$\text{V.A.} : x = 0$$

5.4.2 Homographic functions

Form:

$$f(x) = \frac{ax+b}{cx+d} \qquad (c \neq 0 \wedge a \cdot d \neq b \cdot c)$$

Discuss:

- Function rule: $f(x) = \dfrac{ax+b}{cx+d}$.
- poles f: $-\dfrac{d}{c}$
- dom $f = \mathbb{R} \setminus \{\text{poles}\}$
- zeros f: $\begin{cases} x_0 = -\dfrac{b}{a} & (a \neq 0) \\ \text{none} & (a = 0) \end{cases}$
- V.A.: $x = \text{poles}$
- H.A.: $y = \dfrac{\text{coeff } x}{\text{coeff } x}$
- ran $f = \mathbb{R} \setminus \{\text{H.A.}\}$
- Sign diagram: by splitting numerator and denominator
- Determine intersections
- Sketch graph
- Increasing and decreasing: based on the graph

5.4.3 Rational functions

Form:

$$y = \frac{t(x)}{n(x)}$$

- $t(x)$: numerator, polynomial function
- $n(x)$: denominator, polynomial function

5.5 Exponential Functions

5.5.1 Exponential Functions

Form:

$$f(x) = a^x \qquad (a \in \mathbb{R}_0^+ \setminus \{1\})$$

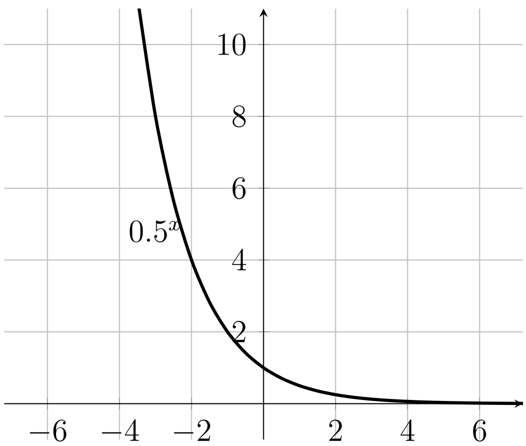
- a: base
- x: exponent

Characteristics

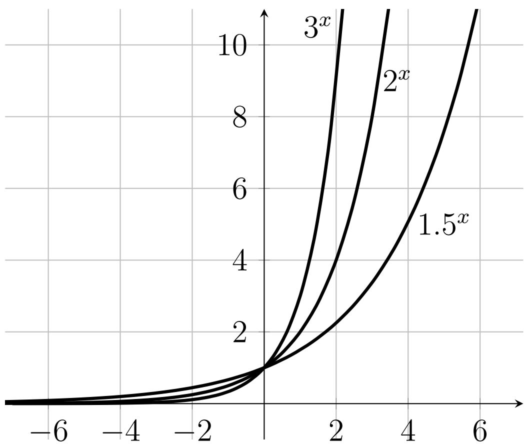
- Domain: $\mathbb{R}$
- Range: $\mathbb{R}_0^+$
- Zeroes: none
- Special points: $(0, 1),\ (1, a)$

Graph

- $0 < a < 1$



- $a > 1$



5.6 Logarithmic Functions

5.6.1 Logarithmic Function

Form:

$$f(x) = \log_a x$$

- a: base $a \in \mathbb{R}_0^+ \setminus \{1\}$
- x: argument

Characteristics:

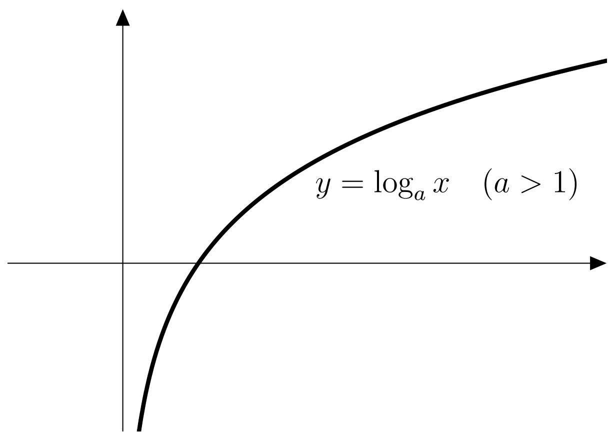
- $\operatorname{dom} \log_a x = \mathbb{R}_0^+$
- $\operatorname{ran} \log_a x = \mathbb{R}$
- Special points: $(1, 0),\ (a, 1)$
- $a > 1$
  - $\log_a x$ is increasing in $\mathbb{R}_0^+$
  - $\lim_{+\infty} \log_a x = +\infty$
  - $\lim_{\substack{x \to 0 \\ >}} \log_a x = -\infty$
- $0 < a < 1$
  - $\log_a x$ is decreasing in $\mathbb{R}_0^+$

  - $\lim_{+\infty} \log_a x = -\infty$
  - $\lim_{\substack{x\to 0\\ >}} \log_a x = +\infty$

- Asymptotes:

  - V.A.: y-axis with equation $x = 0$

Graph:



5.6.2 Main Property of Logarithmic Functions

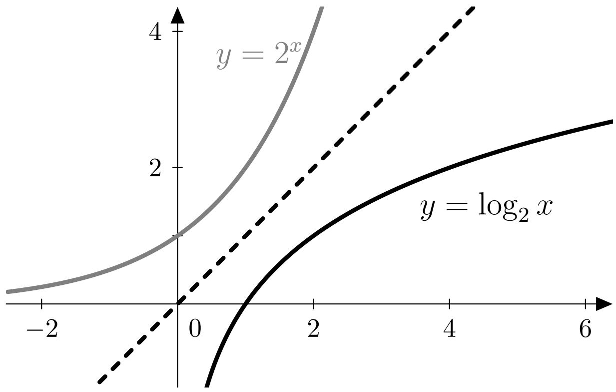
Property: The logarithmic function with base a and the exponential function with base a are inverse functions.

In symbols:

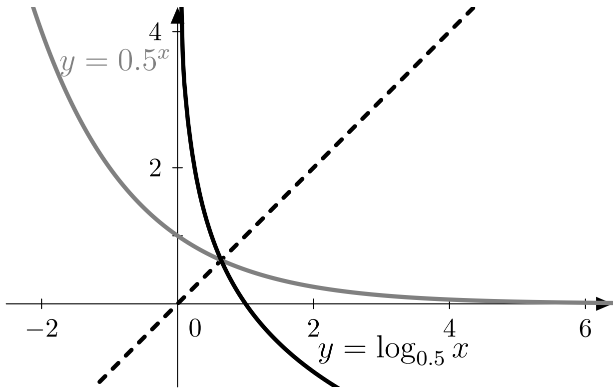
$$f(x) = a^x \Rightarrow f^{-1}(x) = \log_a x$$

Graphically: The graphs of the logarithmic and exponential functions are mirror images of each other with respect to the first bisector in an orthonormal coordinate system.

- $a > 1$



- $0 < a < 1$



5.7 Trigonometric Functions

5.7.1 Sine Function

Form:

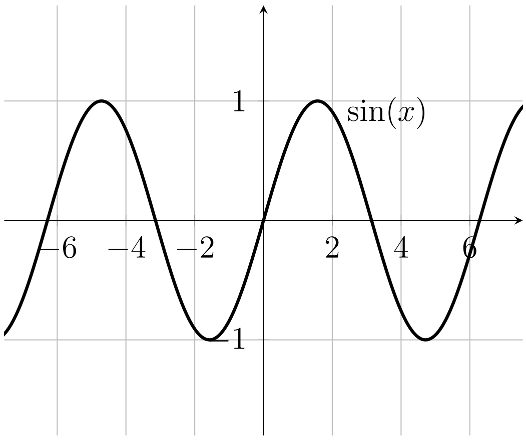
$$f(x) = a\sin(bx + c) + d$$

Characteristics:

- Amplitude: $|a|$
- Period: $T = \dfrac{2\pi}{|b|}$
- Phase Shift: $-\dfrac{c}{b}$
- Vertical Shift: d

- Domain: $\mathbb{R}$
- Range: $[d - |a|, d + |a|]$

Graph:



5.7.2 Cosine Function

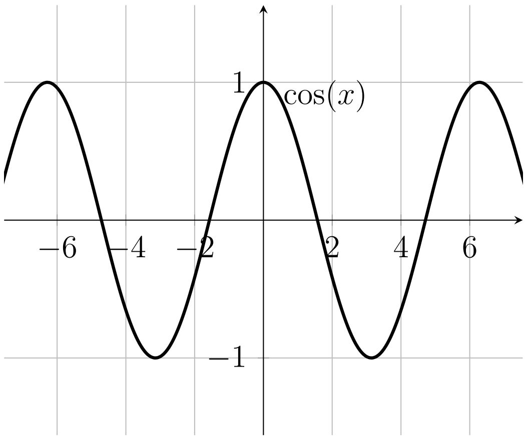
Form:

$$f(x) = a\cos(bx + c) + d$$

Characteristics:

- Amplitude: $|a|$
- Period: $T = \dfrac{2\pi}{|b|}$
- Phase Shift: $-\dfrac{c}{b}$
- Vertical Shift: d
- Domain: $\mathbb{R}$
- Range: $[d - |a|, d + |a|]$

Graph:



5.7.3 Tangent Function

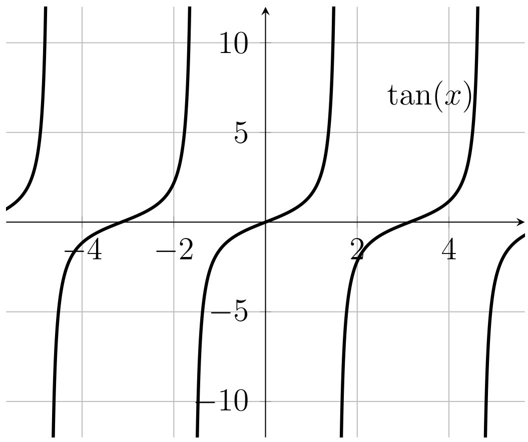
Form:

$$f(x) = a\tan(bx + c) + d$$

Characteristics:

- Period: $T = \dfrac{\pi}{|b|}$
- Phase Shift: $-\dfrac{c}{b}$
- Vertical Shift: d
- Domain: $\mathbb{R} \setminus \left\{\dfrac{\pi}{2} + k\pi : k \in \mathbb{Z}\right\}$
- Range: $(-\infty, +\infty)$

Graph:


10
5
−4
−2
2
4
−5
−10
tan(x)

Chapter 6

Analyse

6.1 Limits

6.1.1 Definition

b is the **limit** of f as x approaches a

$$\begin{aligned}
&\Leftrightarrow \quad \lim_{x\to a} f(x) = b \\
&\Leftrightarrow \quad \forall \epsilon > 0, \exists \delta > 0, \forall x \in \operatorname{dom} f : \\
&\qquad\quad 0 < |x-a| < \delta \Rightarrow |f(x) - b| < \epsilon
\end{aligned}$$

6.1.2 Limit properties

$$\begin{aligned}
\lim_{x\to a} (f(x) \pm g(x)) &= \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) \\
\lim_{x\to a} (c \cdot f(x)) &= c \cdot \lim_{x\to a} f(x) \qquad \text{met } c \in \mathbb{R} \\
\lim_{x\to a} (f(x) \cdot g(x)) &= \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) \\
\lim_{x\to a} \frac{f(x)}{g(x)} &= \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \\
\lim_{x\to a} \left(f(x)^{g(x)}\right) &= \left(\lim_{x\to a} f(x)\right)^{\lim_{x\to a} g(x)} \\
\lim_{x\to a} (f(x)^n) &= \left(\lim_{x\to a} f(x)\right)^n \\
\lim_{x\to a} \sqrt[n]{f(x)} &= \sqrt[n]{\lim_{x\to a} f(x)}
\end{aligned}$$

6.1.3 Calculating infinity

$$
\begin{aligned}
(+\infty) + (+\infty) &= +\infty \\
(-\infty) + (-\infty) &= -\infty \\
k + (+\infty) &= +\infty \\
k + (-\infty) &= -\infty \\
(+\infty) \cdot (+\infty) &= +\infty \\
(+\infty) \cdot (-\infty) &= -\infty \\
(-\infty) \cdot (+\infty) &= -\infty \\
(-\infty) \cdot (-\infty) &= +\infty
\end{aligned}
$$

$$
\begin{aligned}
k \cdot (+\infty) &= \begin{cases} +\infty & (k > 0) \\ -\infty & (k < 0) \end{cases} \\
k \cdot (-\infty) &= \begin{cases} -\infty & (k > 0) \\ +\infty & (k < 0) \end{cases} \\
\frac{k}{+\infty} &= 0 \\
\frac{k}{-\infty} &= 0 \\
\frac{k}{0} &= +\infty \text{ of } -\infty
\end{aligned}
$$

6.1.4 Indeterminate forms

$$(\pm\infty) - (\pm\infty)\ , \quad (\pm\infty) + (\mp\infty)$$

$$\frac{\pm\infty}{\pm\infty}\ , \quad \frac{0}{0}\ , \quad \frac{k}{0}$$

$$0 \cdot (\pm\infty)\ , \quad 1^{\pm\infty}\ , \quad 0^0\ , \quad (\pm\infty)^0$$

6.1.5 Properties of limits

$$\lim_{x\to a} c = c \qquad (c \in \mathbb{R})$$

$$\lim_{x\to\pm\infty} c = c$$

$$\lim_{x\to a} x = a$$

$$\lim_{x\to\pm\infty} x = \pm\infty$$

$$\lim_{x\to+\infty} x^k = +\infty \qquad (k \in \mathbb{N}_0)$$

$$\lim_{x\to-\infty} x^k = \begin{cases} +\infty & (k \text{ is even}) \\ -\infty & (k \text{ is odd}) \end{cases}$$

$$\lim_{x\to\pm\infty} \frac{1}{x} = 0$$

$$\lim_{\substack{x\to 0\\ <}} \frac{1}{x} = -\infty$$

$$\lim_{\substack{x\to 0\\ >}} \frac{1}{x} = +\infty$$

$$\lim_{x\to\pm\infty} \left(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0\right) = \lim_{x\to\pm\infty} a_n x^n$$

6.1.6 Limits of polynomials

$$\lim_{x\to a} f(x) = f(a) \qquad f(x) \text{ is a polynomial and } a \in \mathbb{R}$$

$$\lim_{x\to\pm\infty} \left(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0\right) = \lim_{x\to\pm\infty} a_n x^n$$

6.1.7 Limits of rational functions

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)} \qquad \text{if } g(a) \neq 0$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{k}{0} = \underbrace{+\infty \text{ or } -\infty}_{\text{investigate sign}} \qquad \text{if } f(a) \neq 0 \text{ and } g(a) = 0$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{"0"}{0} = \lim_{x\to a} \frac{(x-a)\ f_r(x)}{(x-a)\underbrace{g_r(x)}_{\text{Horner}}} = \lim_{x\to a} \frac{f_r(x)}{g_r(x)}$$

$$\lim_{x\to\pm\infty} \frac{(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)}{(b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0)} = \lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m}$$

6.1.8 Trig limits

$$\begin{aligned} \lim_{x\to 0} \frac{\sin x}{x} &= 1 \\ \lim_{x\to +\infty} \frac{\sin x}{x} &= 0 \\ \lim_{x\to 0} \frac{\cos x - 1}{x} &= 0 \end{aligned}$$

6.1.9 Special limits

$$\begin{aligned} \lim_{x\to 0} \frac{e^x - 1}{x} &= 1 \\ \lim_{x\to 0} \frac{\ln(x+1)}{x} &= 1 \\ \lim_{n\to +\infty} \left(1 + \frac{1}{n}\right)^n &= e \qquad (n \in \mathbb{N}) \end{aligned}$$

6.1.10 L'Hospital's rule

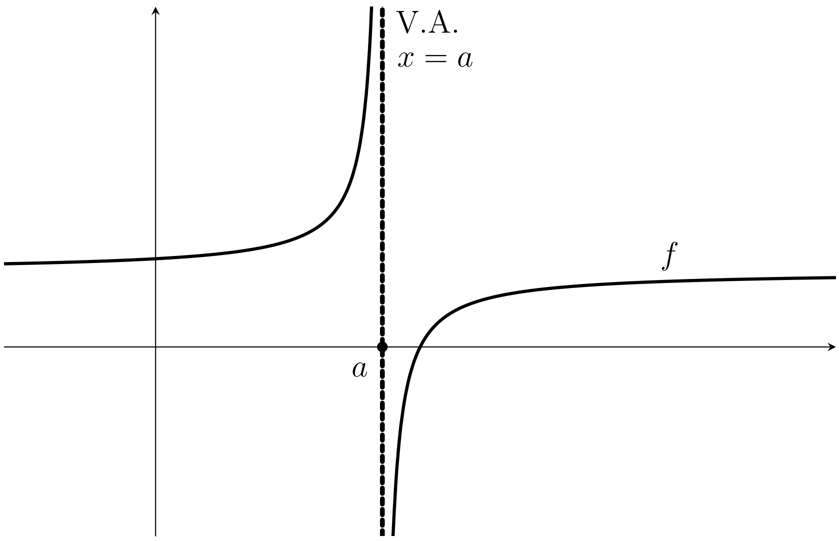
$$\begin{aligned} \lim_{x\to a} \frac{f(x)}{g(x)} &= \frac{"0"}{0} \overset{\text{H}}{=} \lim_{x\to a} \frac{f'(x)}{g'(x)} \\ \lim_{x\to a} \frac{f(x)}{g(x)} &= \frac{"\pm\infty"}{\pm\infty} \overset{\text{H}}{=} \lim_{x\to a} \frac{f'(x)}{g'(x)} \\ \lim_{x\to \pm\infty} \frac{f(x)}{g(x)} &= \frac{"\pm\infty"}{\pm\infty} \overset{\text{H}}{=} \lim_{x\to \pm\infty} \frac{f'(x)}{g'(x)} \end{aligned}$$

6.2 Asymptotes

6.2.1 Vertical asymptotes (V.A.)

Graphical

A **vertical asymptote** of f is a vertical line that comes arbitrarily close to the curve of f.



Algebraic

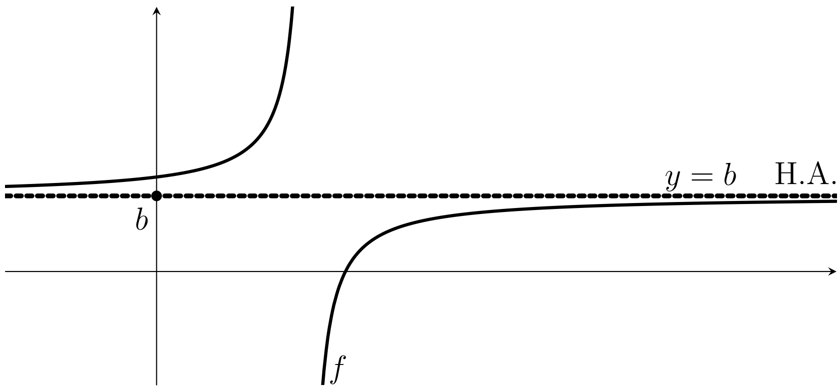
The line $x = a$ is a **vertical asymptote** of f in $a \in \mathbb{R}$

$$\Leftrightarrow \lim_{\substack{x \to a \\ \gtrless}} f(x) = \pm\infty$$

6.2.2 Horizontal asymptotes (H.A.)

Graphical

A **horizontal asymptote** of f is a horizontal line that comes arbitrarily close to the curve of f.



Algebraic

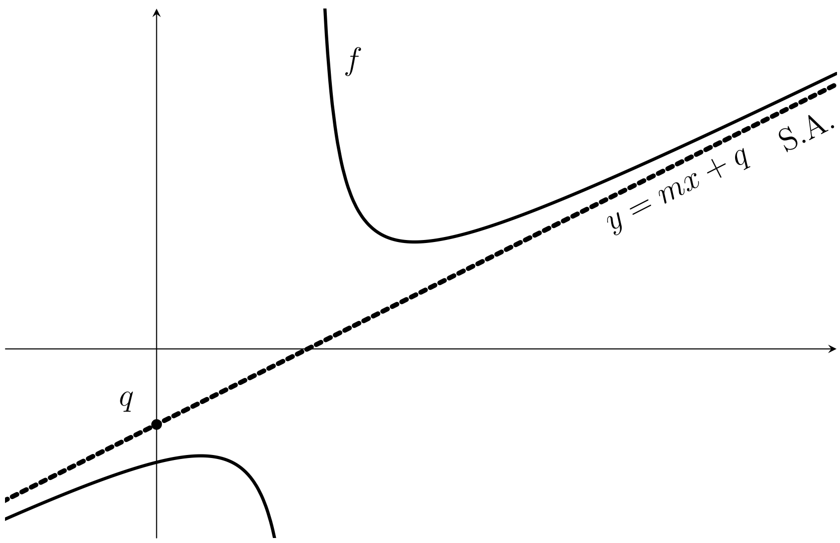
The line $y = b$ is a **horizontal asymptote** of f

$$\Leftrightarrow \lim_{x \to \pm\infty} f(x) = b$$

6.2.3 Oblique asymptotes (S.A.)

Graphical

A **oblique asymptote** of f is a oblique line that comes arbitrarily close to the curve of f.



Algebraïsch

The line $y = mx + q$ is a **oblique asymptote** of f

$$\Leftrightarrow \lim_{x\to\pm\infty} (f(x) - mx - q) = 0$$
$$\Leftrightarrow \begin{cases} m = \lim_{x\to\pm\infty} \dfrac{f(x)}{x} \\ q = \lim_{x\to\pm\infty} (f(x) - mx) \end{cases}$$

6.3 Derivatives

6.3.1 Definition of the derivative

$f'(a)$ is the **derivative** of f with respect to a

$$\Leftrightarrow f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$
$$\Leftrightarrow f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

6.3.2 Base derivatives

$$(c)' = 0$$
$$(x)' = 1$$
$$\left(x^2\right)' = 2x$$
$$\left(x^3\right)' = 3x^2$$
$$(x^n)' = n \cdot x^{n-1}$$
$$\left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}$$

6.3.3 Differentiation of exponential and logaritmic functions

$$(e^x)' = e^x$$
$$(a^x)' = a^x \cdot \ln a$$
$$(\ln x)' = \frac{1}{x}$$
$$(\log_a x)' = \frac{1}{x \cdot \ln a}$$

6.3.4 Differentiation of trigonometric functions

$$(\sin x)' = \cos x$$
$$(\cos x)' = -\sin x$$
$$(\tan x)' = \frac{1}{\cos^2 x} = \sec^2 x$$
$$(\cot x)' = -\frac{1}{\sin^2 x} = -\csc^2 x$$
$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$
$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$
$$(\arctan x)' = \frac{1}{1+x^2}$$
$$(\text{arccot}\ x)' = -\frac{1}{1+x^2}$$

6.3.5 Differentiation formulas

Sum rule

$$(f+g)'(x) = f'(x) + g'(x)$$

Constant multiple rule

$$(c \cdot f)'(x) = c \cdot f'(x)$$

Product rule

$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Quotient rule

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$$

Chain rule

$$(g\,(f\,(x)))' = g'(f(x)) \cdot f'(x)$$

6.3.6 Higher Derivatives

Order	f'	Df	$\frac{\mathrm{d}f}{\mathrm{d}x}$
first derivative	f'	Df	$\frac{\mathrm{d}f}{\mathrm{d}x}$
second derivative	f''	D^2f	$\frac{\mathrm{d}^2f}{\mathrm{d}x^2}$
$\vdots$			
n-th derivative	$f^{(n)}$	D^nf	$\frac{\mathrm{d}^nf}{\mathrm{d}x^n}$

- f' : Lagrange's notatie
- Df : Euler's notatie
- $\frac{\mathrm{d}f}{\mathrm{d}x}$: Leibniz's notatie

6.4 Function Analysis

6.4.1 Tangent Line

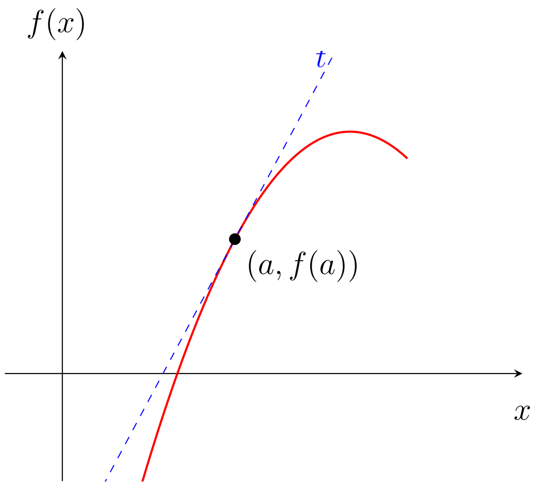
Definition

The **tangent line** to the graph of f at $x = a$:

- point: $(a, f(a))$
- slope: $f'(a)$

- equation:

$$t \leftrightarrow y - f(a) = f'(a)(x-a)$$



6.4.2 Increasing and Decreasing of a Function

Increasing and Decreasing in an Interval

- f is increasing in $[a,b]$ if

$$\forall x_1, x_2 \in [a,b] : x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$$

- f is decreasing in $[a,b]$ if

$$\forall x_1, x_2 \in [a,b] : x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$$

- f is strictly increasing in $[a,b]$ if

$$\forall x_1, x_2 \in [a,b] : x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

- f is strictly decreasing in $[a,b]$ if

$$\forall x_1, x_2 \in [a,b] : x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

Increasing and Decreasing at a Point

- f is increasing at $a \in \text{dom} f$ if

$$\exists B(a,\delta) : f \text{ is increasing in } (a-\delta, a+\delta) \cap \text{dom} f$$

- f is decreasing at $a \in \text{dom} f$ if

$$\exists B(a,\delta) : f \text{ is decreasing in } (a-\delta, a+\delta) \cap \text{dom} f$$

Properties

f is differentiable over $[a, b]$ then $\forall x \in]a, b[$:

- $f'(x) > 0 \Rightarrow f$ is strictly increasing in $[a, b]$
- $f'(x) < 0 \Rightarrow f$ is strictly decreasing in $[a, b]$
- $f'(x) = 0 \Rightarrow f$ is constant in $[a, b]$

f is differentiable at a:

- f is increasing at $a \Leftrightarrow f'(a) \geq 0$
- f is decreasing at $a \Leftrightarrow f'(a) \leq 0$

Rolle's Theorem

$$\begin{cases} f \text{ is continuous in } [a, b] \\ f \text{ is differentiable in } (a, b) \\ f(a) = f(b) \end{cases}$$

$$\Rightarrow \exists c \in (a, b) : f'(c) = 0$$

Lagrange's Mean Value Theorem

$$\begin{cases} f \text{ is continuous in } [a, b] \\ f \text{ is differentiable in } (a, b) \end{cases}$$

$$\Rightarrow \exists c \in]a, b[\colon f'(c) = \frac{f(b) - f(a)}{b - a}$$

6.5 Integrals

6.5.1 Definitions

F is a **primitive function** of f

$$\Leftrightarrow F' = f$$

$$[F(x)]_a^b = F(b) - F(a)$$

6.5.2 Properties of integrals

Integration and differentiation

Inverse operations!

$$\left(\int f(x)\,dx\right)' = f(x)$$
$$\int F'(x)\,dx = F(x) + C$$

Linearity

$$\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$$
$$\int af(x)\,dx = a\int f(x)\,dx$$

6.5.3 Base integrals

$$\int dx = x + C$$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$$

$$\int \frac{1}{x} \, dx = \ln|x| + C$$

$$\int e^x \, dx = e^x + C$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \frac{1}{\cos^2 x} \, dx = \tan x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \frac{1}{\sin^2 x} \, dx = -\cot x + C$$

$$\int \csc^2 x \, dx = -\cot x + C$$

$$\int \frac{1}{1+x^2} \, dx = \arctan x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C$$

$$\int \frac{1}{\sqrt{x^2+k}} \, dx = \ln|x + \sqrt{x^2+k}| + C$$

$$\int \sinh x \, dx = \cosh x + C$$

$$\int \cosh x \, dx = \sinh x + C$$

6.5.4 Integration rules

Substitution rule

$$\int f(t)\,dt = \int f(g(x))g'(x)\,dx$$

with $t = g(x)$

Partial integration

$$\int f'(x) \cdot g(x)\,dx = f(x) \cdot g(x) - \int f(x) \cdot g'(x)\,dx$$

6.5.5 Integration of Rational Functions

Method For integrating a rational function $\frac{P(x)}{Q(x)}$:

1. If $\deg(P(x)) \geq \deg(Q(x))$, perform polynomial long division:

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

   where $\deg(R(x)) < \deg(Q(x))$.

2. Factor the denominator $Q(x)$ into linear factors $(px+q)^m$ and/or irreducible quadratic factors $(ax^2+bx+c)^n$.

3. Decompose $\frac{R(x)}{Q(x)}$ into a sum of partial fractions.

   - For each factor $(px+q)^m$, the decomposition includes:

$$\frac{A_1}{px+q} + \frac{A_2}{(px+q)^2} + \cdots + \frac{A_m}{(px+q)^m}$$

   - For each factor $(ax^2+bx+c)^n$, the decomposition includes:

$$\frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

4. Integrate the polynomial part $S(x)$ and each partial fraction separately.

6.5.6 Applications of Definite Integrals

Area Between a Curve and the x-axis The area A between the graph of $f(x)$ and the x-axis over the interval $[a, b]$:

$$A = \int_a^b |f(x)|\,dx$$

Area Between Two Curves The area A enclosed by the graphs of $f(x)$ and $g(x)$ over the interval $[a, b]$:

$$A = \int_a^b |f(x) - g(x)|\,dx$$

Volume of a Solid of Revolution The volume V of the solid obtained by rotating the graph of $f(x)$ around the x-axis over $[a, b]$:

$$V_x = \pi \int_a^b [f(x)]^2\,dx$$

Arc Length of a Curve The length L of the arc of the graph of $f(x)$ from $x = a$ to $x = b$:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

Chapter 7

Complex Numbers

7.1 Complex Numbers

7.1.1 Definitions of Complex Numbers

Imaginary Unit

- i is the **imaginary unit** with

$$i^2 = -1$$

- Consequence:

$$i^n = i^{n+4} \qquad (n \in \mathbb{N})$$

- $\mathbb{I}$: set of imaginary numbers

$$\mathbb{I} = \{bi \mid b \in \mathbb{R} \wedge i^2 = -1\}$$

Complex Numbers

- A **complex number** has the form

$$z = a + bi \qquad \text{with } a, b \in \mathbb{R}$$

- $a = \mathcal{R}e(z)$: real part
- $b = \mathcal{I}m(z)$: imaginary part
- A complex number is **purely imaginary**

$$\Leftrightarrow a = 0 \wedge b \neq 0$$

- $\mathbb{C}$: set of complex numbers

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R} \wedge i^2 = -1\}$$

- $z_1 = a + bi$ and $z_2 = c + di$ are equal

$$\Leftrightarrow a = c \wedge b = d$$

- z and $\bar{z}$ are each other's complex **conjugates**

$$\Leftrightarrow z = a + bi \;\wedge\; \bar{z} = a - bi$$

7.1.2 Operations on Complex Numbers

Sum (difference)

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

- Sum of imaginary:

$$ai + bi = (a + b)i \in \mathbb{I}$$

- Sum of conjugates:

$$(a + bi) + (a - bi) = 2a \in \mathbb{R}$$

- $\mathbb{C}, +$ is a commutative group
  - Internal
  - Associative
  - Neutral element: 0
  - Symmetric element: $-z$
  - Commutative

Product

$$(a+bi)\cdot(c+di) = (ac-bd)+(ad+bc)i$$

- Product of imaginary:
$$ai\cdot bi = -ab \in \mathbb{R}$$
- Product of conjugates:
$$(a+bi)\cdot(a-bi) = a^2+b^2 \in \mathbb{R}$$
- $\mathbb{C}_0, \cdot$ is a commutative group
  - Internal
  - Associative
  - Neutral element: 1
  - Inverse element:
  $$z^{-1} = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i$$
  - Commutative

Quotient

$$\frac{a+bi}{c+di} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$$

- Practical: Numerator and denominator $\times$ conjugate
- Quotient of imaginary: $\dfrac{ai}{bi} = \dfrac{a}{b} \in \mathbb{R}$
- Quotient of conjugates: $\dfrac{z}{\bar{z}} \in \mathbb{C}$

7.1.3 Powers of Complex Numbers

Definition

$$(a+bi)^n = \underbrace{(a+bi)\cdot(a+bi)\cdot\dots\cdot(a+bi)}_{n\text{ factors}}$$

$$(a+bi)^1 = a+bi$$

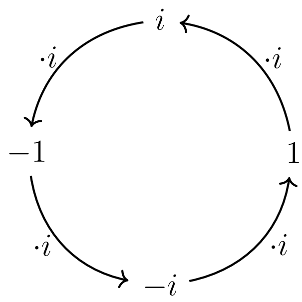
$$(a+bi)^0 = 1$$

$$(a+bi)^{-n} = \frac{1}{(a+bi)^n}$$

Powers of the Imaginary Unit

$$i^{4n} = 1 \qquad i^{4n+1} = i$$

$$i^{4n+2} = -1 \qquad i^{4n+3} = -i$$



Special Powers

$$\begin{aligned}
(a+bi)^0 &= 1 \\
(a+bi)^1 &= a+bi \\
(a+bi)^2 &= a^2 - b^2 + 2abi \\
(a+bi)^3 &= (a^3 - 3ab^2) + (3a^2b - b^3)i
\end{aligned}$$

7.1.4 Square Root of a Complex Number

Definition $x+yi$ is a **square root** of $a+bi$

$$\Leftrightarrow (x+yi)^2 = a+bi$$

General Solution $(b \neq 0)$

$$\begin{aligned}
&(x+yi)^2 = a+bi \\
\Leftrightarrow &\begin{cases} x^2 - y^2 & = a \\ 2xy & = b \end{cases} \\
\Leftrightarrow &\, x+yi = \pm\left(\sqrt{\frac{r+a}{2}} + i\ \mathrm{sgn}(b)\sqrt{\frac{r-a}{2}}\right) \\
&\text{with } r = \sqrt{a^2+b^2}
\end{aligned}$$

Solutions of Real Numbers $(b = 0)$

$$a \geq 0: \qquad x + yi = \pm\sqrt{a}$$
$$a < 0: \qquad x + yi = \pm i\sqrt{-a}$$

7.1.5 Complex Quadratic Equations

Standard Form Form of **quadratic equation** in $\mathbb{C}$ is

$$az^2 + bz + c = 0 \text{ with } a \in \mathbb{C}_0, b, c \in \mathbb{C}$$

Properties

- Every quadratic equation in $\mathbb{C}$ has two solutions in $\mathbb{C}$, counted with multiplicity.
- If $a, b, c \in \mathbb{R}$, the solutions are either both real or a pair of non-real complex conjugates.

Solution Method

$$az^2 + bz + c = 0$$

- Calculate $D = b^2 - 4ac$
- Calculate $\pm\sqrt{D}$
- Calculate $z_{1,2} = \dfrac{-b \pm \sqrt{D}}{2a}$

7.1.6 Binomial Equations

Definition Form of a **binomial equation** in the unknown z is

$$z^n = a \text{ with } a \in \mathbb{C}_0, n \in \mathbb{N}_0$$

Properties

- A binomial equation has n different complex solutions.
- A binomial equation with $a \in \mathbb{R}$ has at most 2 real solutions.

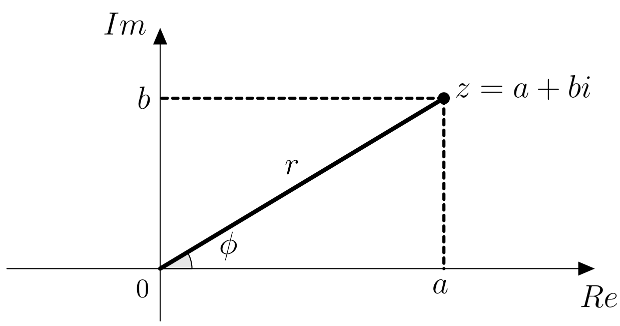
7.2 Polar Form

7.2.1 Polar Form

Definition

$$z = \underbrace{a+bi}_{\text{Cartesian}} = \underbrace{r(\cos\phi + i\sin\phi)}_{\text{Polar}}$$

- **modulus** $r = |z|$ is the distance from the origin to the complex number
- For $z \neq 0$, the **principal argument** $\phi = \arg z \in [0, 360°)$ is the angle between the positive real axis and the half-line from the origin through z
- $\begin{cases} r = \sqrt{a^2+b^2} & \\ \tan\phi = \dfrac{b}{a} & (a \neq 0) \\ \phi = 90° \ \vee \ 270° & (a = 0) \end{cases}$
- Choose ϕ in the quadrant determined by $z = a + bi$.
- $\begin{cases} a &= r \cdot \cos\phi \\ b &= r \cdot \sin\phi \end{cases}$



Properties

- $z_1 = z_2 \Leftrightarrow r_1 = r_2 \wedge \phi_1 = \phi_2$
- $r \in \mathbb{R}^+$
- $z = 0 \Rightarrow r = 0$ and ϕ is undetermined
- $|z| = |\overline{z}| \wedge \arg z \equiv -\arg \bar{z} \pmod{360°}$
- $|z| = |-z| \wedge \arg(-z) \equiv \arg z + 180° \pmod{360°}$

Special Cases

z	$r = \|z\|$	$\phi = \arg z$
1	1	$0°$
i	1	$90°$
-1	1	$180°$
$-i$	1	$270°$

7.2.2 Operations in Polar Form

Product

$$z_1 \cdot z_2 = r_1 r_2 \left(\cos(\phi_1 + \phi_2) + i \sin(\phi_1 + \phi_2)\right)$$

Quotient

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \left(\cos(\phi_1 - \phi_2) + i \sin(\phi_1 - \phi_2)\right)$$

Power

$$z^n = r^n \left(\cos(n\phi) + i \sin(n\phi)\right)$$

7.2.3 Euler's Formula

$$e^{i\phi} = \cos\phi + i \sin\phi$$

7.2.4 de Moivre's Formula

$$(\cos\phi + i \sin\phi)^n = \cos(n\phi) + i \sin(n\phi)$$

7.3 Polynomials with Complex Coefficients

7.3.1 Definitions

Polynomial in one variable z

$$\begin{aligned} A(z) &= \sum_{i=0}^{n} a_i z^i \\ &= a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \end{aligned}$$

- Coefficients: $a_n, a_{n-1}, ..., a_1, a_0 \in \mathbb{C}$
- Degree: $\deg(A) = n \in \mathbb{N}$

- Notation: $A(z), B(z), ...$
- Set: $\mathbb{C}[z]$
- Numeric value $A(z)$ for $\omega = a + bi$: $A(\omega)$
- Root ω of $A(z) \Leftrightarrow A(\omega) = 0$

7.3.2 Divisibility

Definition

$A(z)$ is **divisible** by $D(z)$

$$\begin{aligned} &\Leftrightarrow D(z)|A(z) \\ &\Leftrightarrow \exists Q(z) \in \mathbb{C}[z] : A(z) = D(z) \cdot Q(z) \\ &\Leftrightarrow R(z) = 0 \end{aligned}$$

- $A(z)$: dividend
- $D(z)$: divisor
- $Q(z)$: quotient
- $R(z)$: remainder

Criterion for divisibility

$A(z)$ is divisible by $z - \omega$

$$\begin{aligned} &\Leftrightarrow z - \omega|A(z) \\ &\Leftrightarrow A(\omega) = 0 \end{aligned}$$

Product Divisibility

If $\omega_1, \omega_2 \in \mathbb{C}$ and $\omega_1 \neq \omega_2$ then

$$((z - \omega_1) \cdot (z - \omega 2))\,|A(z)$$

$$\Leftrightarrow (z - \omega_1)|A(z) \wedge (z - \omega_2)|A(z)$$

D'Alembert's Theorem

(at least one root)

$$\begin{aligned} &A(z) \in \mathbb{C}[z] \wedge \deg\left(A(z)\right) \geq 1 \\ &\qquad \Longrightarrow \exists \omega \in \mathbb{C} : A(\omega) = 0 \end{aligned}$$

Number of zeros

A polynomial of degree n has exactly n (possibly coinciding) roots in $\mathbb{C}$

Factorization

$$\begin{aligned} A(z) &= a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \\ &= a_n (z - \omega_1)(z - \omega_2) \cdots (z - \omega_n) \end{aligned}$$

- ω_i : roots of $A(z)$

Chapter 8

Statistics

8.1 Descriptive statistics

8.1.1 Definitions

- **Population** : set of similar items
- **Sample** : subset of the population
- **Sample size** n : total number of measurements
- **Variable** x : measure

8.1.2 Variables

- Continuous: gaussian distribution
- Discrete: ordinal, nominal or binomial

8.1.3 Measures

Central tendency

- mean $\bar{x}$

$$\bar{x} = \frac{1}{n} \sum_i x_i$$

- median Med : middle
- mode Mod : most occurring

Dispersion

- range Ran : difference between the largest and smallest values

$$Ran = x_{\max} - x_{\min}$$

- variance Var :

$$Var_x = \frac{1}{n} \sum_i (x_i - \bar{x})^2$$

- standard deviation s :

$$s_x = \sqrt{\frac{1}{n} \sum_i (x_i - \bar{x})^2}$$

8.2 Combinatorics

8.2.1 Product rule

r independent partial decisions with

$$n_1, n_2, \ldots, n_r$$

possibilities

$$\Rightarrow \#\text{possibilities} = n_1 \cdot n_2 \cdot \ldots \cdot n_r$$

8.2.2 Sum rule

With double counting

$$\#(A \cup B) = \#A + \#B - \#(A \cap B)$$

- $\#A$: number of elements in A
- $A \cup B$: union, A *or* B
- $A \cap B$: intersection, A *and* B

Mutually exclusive possibilities If A and B are disjoint sets

$$\Rightarrow \#(A \cup B) = \#A + \#B$$

8.2.3 Factorials

Definition

$$\begin{aligned}\forall n \in \mathbb{N}\backslash\{0,1\} : n! &= n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1 \\ 1! &= 1 \\ 0! &= 1\end{aligned}$$

Properties

$$\begin{aligned}(n+1)! &= (n+1) \cdot n! \\ n! &= n \cdot (n-1)!\end{aligned}$$

8.2.4 Permutations

$$\begin{aligned}P_n &= n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 \\ &= n!\end{aligned}$$

8.2.5 K-Permutations

$$\begin{aligned}P_n^k &= n \cdot (n-1) \cdot (n-2) \cdots (n-k+1) \\ &= \frac{n!}{(n-k)!}\end{aligned}$$

8.2.6 K-Permutations with repetitions

$$\bar{V}_n^k = n^k$$

8.2.7 Combinations

$$C_n^k = \frac{n!}{k! \cdot (n-k)!}$$

8.2.8 Combinations with repetitions

$$\bar{C}_n^k = C_{n+k-1}^k$$

8.2.9 Anagrams

$$
\begin{aligned}
P_n^{n_1,n_2,n_3,\ldots,n_k} &= \frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdot \cdots \cdot n_k!} \\
&= \frac{\left(\sum_{i=1}^{k} n_i\right)!}{\prod_{i=1}^{k} n_i!}
\end{aligned}
$$

8.2.10 Pascal's triangle

Binomialcoefficients

the **binomiaal coefficient** where from n elements we choose k:

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$$

Recursive formula

If $k, n \in \mathbb{N}$ and $k < n$:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Triangle

$n \downarrow$														
0	1													
1	1	1												
2	1	2	1											
3	1	3	3	1										
4	1	4	6	4	1									
5	1	5	10	10	5	1								
6	1	6	15	20	15	6	1							
7	1	7	21	35	35	21	7	1						
8	1	8	28	56	70	56	28	8	1					
9	1	9	36	84	126	126	84	36	9	1				
10	1	10	45	120	210	252	210	120	45	10	1			
11	1	11	55	165	330	462	462	330	165	55	11	1		
12	1	12	66	220	495	792	924	792	495	220	66	12	1	
	0	1	2	3	4	5	6	7	8	9	10	11	12	$\leftarrow k$

Properties

$$\binom{n}{k} = \binom{n}{n-k}$$

Binomium of Newton

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Chapter 9

Physics

9.1 Optics

9.1.1 Snell's Law

$$\frac{\sin i}{\sin r} = n$$

- i: angle of incidence relative to the normal
- r: angle of refraction relative to the normal
- n: refractive index of the medium

9.1.2 Lens Formula

$$\frac{1}{v} + \frac{1}{b} = \frac{1}{f}$$

- v: image distance
- b: object distance
- f: focal length of the lens

9.2 Pressure

9.2.1 Hydrostatic Pressure

$$p = \rho \cdot g \cdot h + p_0$$

- p: pressure at a certain depth

- ρ: density of the fluid
- g: gravitational acceleration
- h: height/depth in the fluid
- p_0: pressure at the surface (usually atmospheric pressure)

9.2.2 Buoyant Force in a Fluid

$$F_A = \rho \cdot g \cdot V$$

- F_A: buoyant force
- ρ: density of the fluid
- g: gravitational acceleration
- V: volume of the displaced fluid

9.3 Gas Laws

9.3.1 Ideal Gas Law

$$PV = nRT$$

- P: pressure of the gas
- V: volume of the gas
- n: number of moles of gas
- R: universal gas constant
- T: temperature in Kelvin

9.3.2 Boyle's Law

$$P_1V_1 = P_2V_2$$

- P_1: initial pressure
- V_1: initial volume
- P_2: final pressure
- V_2: final volume

9.4 Thermodynamics

9.4.1 First Law of Thermodynamics

$$\Delta U = Q - W$$

- ΔU: change in internal energy
- Q: heat added
- W: work done by the system

9.4.2 Heat Transfer

$$Q = mc\Delta T$$

- Q: heat transferred
- m: mass of the object
- c: specific heat capacity of the material
- ΔT: change in temperature

9.4.3 Heat Capacity

$$Q = C\Delta\theta$$

- Q: amount of heat
- C: heat capacity of the object
- $\Delta\theta$: change in temperature

9.4.4 Heat of Fusion

$$Q = \ell_s \cdot m$$

- Q: amount of heat required for melting
- ℓ_s: latent heat of fusion of the material (heat per unit mass)
- m: mass of the material

9.4.5 Heat of Vaporization

$$Q = \ell_v \cdot m$$

- Q: amount of heat required for vaporization
- ℓ_v: latent heat of vaporization of the material (heat per unit mass)
- m: mass of the material

9.5 Electrostatics

9.5.1 Coulomb's Law

$$F = k\frac{|q_1 q_2|}{r^2}$$

- F: force between the two charges
- q_1 and q_2: values of the two charges
- r: distance between the centers of the two charges
- k: electrostatic constant

  $k \approx 8.9875 \times 10^9 \text{ N m}^2/\text{C}^2$

9.5.2 Electric Field Due to a Point Charge

$$E = k\frac{|q|}{r^2}$$

- E: electric field strength at a point
- r: distance from the charge
- q: charge

9.5.3 Electric Potential Due to a Point Charge

$$V = k\frac{q}{r}$$

- V: electric potential at a point
- r: distance from the charge
- q: charge

9.5.4 Relationship Between Electric Field and Potential

$$E = -\nabla V$$

9.5.5 Capacitance of a Capacitor

$$C = \frac{\varepsilon_0 \varepsilon_r A}{d}$$

- C: capacitance
- ε_0: vacuum permittivity
- ε_r: relative permittivity (dielectric constant) of the material between the plates
- A: area of one of the plates of the capacitor
- d: distance between the plates of the capacitor

9.5.6 Energy Stored in a Capacitor

$$U = \frac{1}{2} C V^2$$

9.5.7 Continuity Equation (Gauss' Law for Electricity)

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enc}}}{\varepsilon_0}$$

9.6 Electromagnetism

9.6.1 Lorentz force on a moving charge in a magnetic field

$$F_L = B \cdot |Q| \cdot v$$

- F_L: Lorentz force
- B: magnetic field strength
- Q: charge
- v: velocity of the charge perpendicular to the magnetic field

9.6.2 Lorentz force on a current-carrying conductor in a magnetic field

$$F_L = B \cdot I \cdot \ell$$

- F_L: Lorentz force
- B: magnetic field strength
- I: current through the conductor
- ℓ: length of the segment of the conductor in the magnetic field

9.6.3 Magnetic field around a long straight current-carrying wire

$$B = \frac{\mu_0 \cdot I}{2\pi r}$$

- B: magnetic field strength at a distance r from the wire
- μ_0: magnetic permeability of vacuum
- I: current through the wire
- r: radial distance from the point of interest to the wire

9.6.4 Magnetic field in the core of a long solenoid

$$B = \frac{\mu_0 \cdot N \cdot I}{\ell}$$

- B: magnetic field strength in the core of the solenoid
- μ_0: magnetic permeability of vacuum
- N: number of turns of the solenoid
- I: current through the solenoid
- ℓ: length of the solenoid

9.6.5 Magnetic flux

$$\Phi = B \cdot A$$

- Φ: magnetic flux through a surface
- B: magnetic field strength perpendicular to the surface
- A: area through which the magnetic field lines pass

9.6.6 Biot-Savart law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{I\, d\mathbf{l} \times \hat{r}}{r^2}$$

- $\mathbf{B}$: magnetic field at position $\mathbf{r}$
- I: current through the infinitesimally small wire element $d\mathbf{l}$
- $\hat{r}$: unit vector pointing from the wire element to the observation location
- r: distance from the wire element to the observation location
- μ_0: magnetic permeability of vacuum

9.6.7 Ampère's law

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$

- $\mathbf{B}$: magnetic field
- I_{enc}: current enclosed by the loop
- μ_0: magnetic permeability of vacuum

9.6.8 Faraday's law of electromagnetic induction

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

- $\mathcal{E}$: Induced electromotive force (EMF)
- Φ_B: Magnetic flux

9.6.9 Lenz's law

The direction of the induced current (or EMF) is such that it opposes the change that caused it.

9.6.10 Maxwell's equations

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}
\end{aligned}$$

- $\mathbf{E}$: electric field
- $\mathbf{B}$: magnetic field
- ρ: electric charge density
- $\mathbf{J}$: current density
- ε_0: electric permittivity of vacuum
- μ_0: magnetic permeability of vacuum

9.6.11 Speed of light in vacuum

$$\begin{aligned}
c &= \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \\
&\approx 3.00 \times 10^8 \,\mathrm{m/s}
\end{aligned}$$

- c: speed of light in vacuum
- μ_0: magnetic permeability of vacuum
- ε_0: electric permittivity of vacuum

9.7 Nuclear Physics

9.7.1 Activity of a Radioactive Substance

$$A(t) = \lambda \cdot N(t)$$

- $A(t)$: activity (number of decays per unit of time) at time t
- λ: decay constant of the substance
- $N(t)$: number of undecayed nuclei at time t

9.7.2 Number of Remaining Nuclei After Time t

$$N(t) = N_0 \cdot e^{-\lambda t}$$

- $N(t)$: number of undecayed nuclei at time t
- N_0: initial number of nuclei at time $t = 0$
- λ: decay constant of the substance
- t: time elapsed since the beginning of the observation

9.7.3 Relationship of the Decay Constant with the Half-life

$$\lambda = \frac{0.693}{T_{1/2}}$$

- λ: decay constant of the substance
- $T_{1/2}$: half-life of the radioactive substance (the time in which half of the nuclei decay)

9.8 Kinematics

9.8.1 Displacement with Constant Velocity

$$x = x_0 + v_x \cdot t$$

- x: final position at time t
- x_0: initial position
- v_x: velocity in the x-direction (assumed constant)
- t: elapsed time

9.8.2 Velocity with a Constant Acceleration

$$v_x = v_{x,0} + a_x \cdot t$$

- v_x: velocity in the x-direction at time t
- $v_{x,0}$: initial velocity in the x-direction
- a_x: acceleration in the x-direction (assumed constant)
- t: elapsed time

9.8.3 Displacement with a Constant Acceleration

$$x = x_0 + v_{x,0} \cdot t + \frac{a_x \cdot t^2}{2}$$

- x: final position at time t
- x_0: initial position
- $v_{x,0}$: initial velocity in the x-direction
- a_x: acceleration in the x-direction (assumed constant)
- t: elapsed time

9.9 Dynamics

9.9.1 Newton's Second Law

$$F = m \cdot a$$

- F: net force acting on an object
- m: mass of the object
- a: acceleration of the object

9.9.2 Newton's Law of Universal Gravitation

Gravitational force between two masses:

$$F = G \cdot \frac{m_1 \cdot m_2}{r^2}$$

- F: gravitational force between two masses
- G: gravitational constant
- m_1 and m_2: masses of the two objects
- r: distance between the centers of the two masses

9.9.3 Weight of an Object Near the Earth's Surface

$$F_z = m \cdot g$$

- F_z: gravitational force acting on an object
- m: mass of the object
- g: acceleration due to gravity (on Earth approximately $9.81\,\mathrm{m/s^2}$)

9.9.4 Work Done by a Force

$$W = F \cdot d$$

- W: work
- F: force
- d: displacement in the direction of the force

9.9.5 Power

$$P = \frac{W}{t}$$

- P: power
- W: work
- t: time

9.9.6 Spring Force

$$F_v = k \cdot |\Delta \ell|$$

- F_v: spring force
- k: spring constant
- $\Delta \ell$: extension or compression of the spring from its equilibrium position

9.9.7 Kinetic Energy

$$E_k = \frac{m \cdot v^2}{2}$$

- E_k: kinetic energy
- m: mass of the object
- v: velocity of the object

9.9.8 Potential Energy in a Gravitational Field

$$E_p = m \cdot g \cdot h$$

- E_p: potential energy
- m: mass of the object
- g: acceleration due to gravity
- h: height above a reference point

9.9.9 Potential Energy in a Spring

$$E_p = \frac{k \cdot x^2}{2}$$

- E_p: potential energy of an extended or compressed spring
- k: spring constant
- x: extension or compression of the spring from its equilibrium position

9.9.10 Centripetal Acceleration

$$a_n = \frac{v^2}{r}$$

- a_n: centripetal acceleration
- v: velocity of the object
- r: radius of the circular motion

9.10 Oscillations and Waves

9.10.1 Simple Harmonic Oscillation

$$y(t) = A \cdot \sin(\omega t + \phi)$$

- $y(t)$: displacement of the object at time t
- A: amplitude of the oscillation
- ω: angular frequency
- t: time
- ϕ: phase angle

9.10.2 Wave Equation

$$y(x, t) = A \cdot \sin(kx - \omega t + \phi)$$

- $y(x, t)$: displacement of the point at position x at time t
- A: amplitude of the wave
- k: wave number
- ω: angular frequency
- x: position
- t: time
- ϕ: phase angle

9.10.3 Relation between Angular Frequency and Period

$$\omega = \frac{2\pi}{T}$$

- ω: angular frequency
- T: period

9.10.4 Relation between Wavelength, Speed, and Frequency

$$\lambda = \frac{v}{f} = v \cdot T$$

- λ: wavelength
- v: wave speed
- f: frequency of the wave
- T: period of the wave

9.10.5 Period of a Simple Harmonic Oscillation

$$T = 2\pi\sqrt{\frac{m}{k}}$$

- T: period of the oscillation
- m: mass of the oscillating object
- k: spring constant

9.10.6 Harmonic Frequencies in a Closed Pipe

$$f_n = n \cdot \frac{v}{4L}$$

- f_n: n-th harmonic frequency
- n: harmonic number (1, 2, 3, ...)
- v: speed of sound in the pipe
- L: length of the pipe

9.11 Sound

9.11.1 Speed of Sound in a Medium

$$v = \sqrt{\frac{B}{\rho}}$$

- v: speed of sound in the medium
- B: bulk modulus of the medium
- ρ: density of the medium

9.11.2 Doppler Effect for a Stationary Observer

$$f' = f\left(\frac{c}{c - v_s}\right)$$

- f': observed frequency
- f: emitted frequency
- c: speed of sound in the medium
- v_s: speed of the sound source (positive when approaching the observer, negative when receding)

9.11.3 Sound Intensity

$$I = \frac{P}{A}$$

- I: sound intensity
- P: sound power
- A: area over which the sound spreads

9.11.4 Sound Level

$$L = 10 \cdot \log_{10}\left(\frac{I}{I_0}\right)$$

- L: sound level in decibels (dB)
- I: sound intensity
- I_0: reference intensity (typically $1 \times 10^{-12}\mathrm{W/m^2}$)

9.12 Physical Constants

9.12.1 Atmospheric Pressure

$$p_{\text{atm}} = 1.00 \times 10^5\,\text{Pa}$$

9.12.2 Gravitation

Gravitational Constant

$$G = 6.67430 \times 10^{-11}\,\text{m}^3\text{kg}^{-1}\text{s}^{-2}$$

Acceleration Due to Gravity near Earth's Surface

$$g = 9.81\,\text{m/s}^2$$

* Varies depending on location

9.12.3 Speed of Light in Vacuum

$$c = 2.99792458 \times 10^8\,\text{m/s}$$

9.12.4 Planck's Constant

$$h = 6.62607015 \times 10^{-34}\,\text{J} \cdot \text{s}$$

9.12.5 Gas Constant

$$R = 8.314462618\,\text{J} \cdot \text{mol}^{-1} \cdot \text{K}^{-1}$$

9.12.6 Charge of an Electron

$$e = 1.602176634 \times 10^{-19}\,\text{C}$$

9.12.7 Density

Material	**Density**
Water at 20°C	$\rho_{\text{water}} = 998.2\,\text{kg/m}^3$
Lead	$\rho_{\text{lead}} = 11\,342\,\text{kg/m}^3$
Gold	$\rho_{\text{gold}} = 19\,320\,\text{kg/m}^3$
Aluminum	$\rho_{\text{aluminum}} = 2\,700\,\text{kg/m}^3$
Iron	$\rho_{\text{iron}} = 7\,874\,\text{kg/m}^3$

9.12.8 Thermal Properties

Property	**Value**
Specific heat of water	$c_{\mathrm{water}} = 4.186\,\mathrm{J/g\cdot K}$
Heat of vaporization of water	$\ell_{\mathrm{v}} = 2.260\,\mathrm{kJ/g}$
Heat of fusion of ice	$\ell_{\mathrm{s}} = 334\,\mathrm{J/g}$

9.12.9 Electrostatics

Property	**Value**
Coulomb's Constant	$k_e = \frac{1}{4\pi\epsilon_0}$
Magnetic Permeability of Vacuum	$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{T\cdot m/A}$
Electric Permittivity of Vacuum	$\epsilon_0 = 8.854 \times 10^{-12}\,\mathrm{C^2/N\cdot m^2}$
Elementary Charge	$e = 1.602 \times 10^{-19}\,\mathrm{C}$

9.12.10 Energy

$$1\,\mathrm{kWh} = 3.6 \times 10^{6}\,\mathrm{J}$$
$$1\,\mathrm{eV} = 1.602 \times 10^{-19}\,\mathrm{J}$$

9.12.11 Masses of Fundamental Particles

Particle	**Mass**
Electron	$m_e = 9.109 \times 10^{-31}\,\mathrm{kg}$
Proton	$m_p = 1.673 \times 10^{-27}\,\mathrm{kg}$
Neutron	$m_n = 1.675 \times 10^{-27}\,\mathrm{kg}$

Avogadro's Number

$$N_A = 6.022 \times 10^{23}\,\mathrm{mol^{-1}}$$

Chapter 10

Symbols

10.1 Greek symbols

10.1.1 Greek symbols

symbol	**small**	**capital**
alpha	α	A
beta	β	B
gamma	γ	Γ
delta	δ	Δ
e-psilon	ϵ	E
zeta	ζ	Z
eta	η	H
theta	θ or ϑ	Θ
jota	ι	I
kappa	κ	K
lambda	λ	Λ
mu	μ	M
nu	ν	N
ksi	ξ	Ξ
o-mikron	o	O
pi	π or ϖ	Π
ro	ρ or ϱ	P
sigma	σ or ς	Σ
tau	τ	T
u-psilon	υ	Υ
fi	ϕ or φ	Φ
chi	χ	X
psi	ψ	Ψ
o-mega	ω	Ω

10.2 Logic Symbols

10.2.1 Propositional calculus

symbol	name	description
$\Rightarrow$	implication	if ... then also ...
		implies
$\Leftrightarrow$	equivalence	is equivalent with
$\wedge$	conjunction	and
$\vee$	disjunction	or
$\dot{\vee}$	excluding or	either
$\neg$	negation	not

10.2.2 Predicate logic

symbol	name	discription
$\forall$	universal quantification	for all
$\exists$	existential quantification	there exists
$\exists!$		there is precisely one
$\nexists$		for no

10.3 Sets

10.3.1 Number systems

set	symbol
natural numbers	$\mathbb{N}$
integers	$\mathbb{Z}$
rational numbers	$\mathbb{Q}$
real numbers	$\mathbb{R}$
complex numbers	$\mathbb{C}$
quaternions	$\mathbb{H}$

Remark: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$

10.3.2 Set theory

symbol	**name**	**definition**
$\{...\}$	set	the collection of ...
$\{...\|...\}$	set	the set of ... such that ...
$\emptyset$ of $\{\}$	empty	the empty set
$\in$	membership	element in
$\subseteq$	subset	subset of
$\cup$	union	union of ... and ...
$\cap$	intersection	intersection of ... and ...
$\setminus$	set difference	minus
$\times$	cartesian product	times
2^X	power set	set of all subsets
$\#$	cardinality	number of elements

10.4 Mathematical constants

10.4.1 Pi

Ratio of a circle's circumference to the diameter

$$\pi = 3.14159\,26535\,89793\,23846\,...$$

10.4.2 Euler's number

The limit $e = \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n$

$$e = 2.71828\,18284\,59045\,23536\,...$$

10.4.3 Pythagoras constant

Positive root of $x^2 = 2$

$$\sqrt{2} = 1.41421\,35623\,73095\,04880\,...$$

10.4.4 Theodorus constant

Positive root of $x^2 = 3$

$$\sqrt{3} = 1.73205\,08075\,68877\,29352\,...$$

10.4.5 Golden ratio (Phi)

The ratio of two numbers such that the ratio of their sum to the larger of the two numbers is the same, thus $\varphi = \frac{a}{b} = \frac{a+b}{a} = \frac{1+\sqrt{5}}{2}$

$$\varphi = 1.61803\,39887\,49894\,84820\,...$$

10.4.6 Imaginary unit

The number $i \in \mathbb{C}$ is any root of $x^2 = -1$.

Chapter 11

Tables

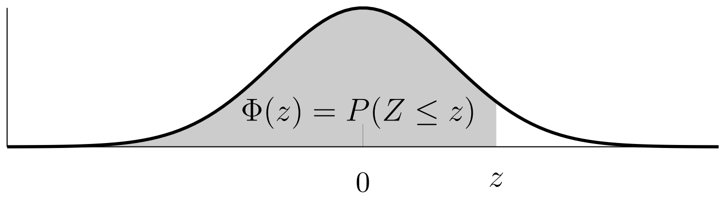
11.1 Powers

11.1.1 Powers with a natural base and natural exponent

a^n	2	3	4	5	6	7	8	9	10
2	4	8	16	32	64	128	256	512	1024
3	9	27	81	243	729				
4	16	64	256	1024					
5	25	125	625						
6	36	216							
7	49	343							
8	64	512							
9	81	729							
10	100								
11	121								
12	144								
13	169								
14	196								
15	225								
16	256								
17	289								
18	324								
19	361								
20	400								

11.2 Standard normal distribution

11.2.1 Standard normal distribution



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000